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The Limit Assumption*

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Abstract In the literature on modality and conditionals, the Limit Assumption is routinely invoked to ensure that a simple definition of necessity (truth at all minimal worlds) can safely be substituted for a more complicated one (cf. Lewis's and Kratzer's definitions involving multiple layers of quantification). The Limit Assumption itself was formulated by David Lewis in 1973 and 1981, and while its plausibility has at times been debated on philosophical grounds, its content is rarely questioned. I show that there is in fact no single "correct" Limit Assumption: which one is right depends on structural properties of the model and the intended notion of necessity. The version that is most widely appealed to in the linguistic literature turns out to be incorrect for its intended purpose. The source of the confusion can be traced back to Lewis himself.

Keywords: conditionals, modality, ordering semantics, limit assumption

1 Kratzer's ordering semantics

The standard approach in linguistics to the formal semantic analysis of modality and conditionals goes back to Kratzer's (1981, 1991) seminal work, which was in turn inspired by work in philosophical logic, in particular the writings on counterfactuals by Lewis (1973) and others.

1.1 Modal base and ordering source

The sentences in (1) are analyzed in terms of the modal operators '*must*' and '*may*' taking the sentence radical '*John home*' as their prejacent, as indicated schematically on the right.

(1) a. John must be at home. *must*(*John home*)

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b. John may be at home. *may*(*John home*)

Such sentences are taken to assert that their prejacent is either a consequence of (in the case of '*must*') or consistent with (in the case of '*may*') a body of premises or background information. On any particular occasion of use, the relevant body of information is specified by two contextually given parameters: the *modal base* furnishes information that is taken to be firmly established in the relevant sense (e.g., known in the epistemic case, or true for a circumstantial reading), whereas the *ordering source* contributes defeasible information about such notions as normalcy or stereotypicality, preferences, obligations, and the like.

Formally, both the modal base and the ordering source are modeled as *conversational backgrounds* — functions from possible worlds to sets of propositions. Propositions are in turn modeled as sets of possible worlds, thus a conversational background is a function from possible worlds to sets of sets of possible worlds.

The interpretation of modal and conditional sentences relative to modal bases and ordering sources has been formalized in two principal ways, known as *Premise Semantics* and *Ordering Semantics* (Lewis 1981). In this paper I focus on the Ordering Semantic approach. Following much of the linguistic literature, I reserve the variables f and g for the modal base and ordering source, respectively.

Kratzer (1981) noted that each modal base uniquely determines an accessibility relation between possible worlds: For an arbitrary modal base f, let $wR^f v$ iff $v \in \cap f(w)$. The set R^f_w of worlds accessible from w (i.e., at which all propositions in $f(w)$ are true) plays an important role in Kratzer's semantics, yet there is no established name for it. I follow Cariani et al. (2013), Kaufmann & Kaufmann (2015) and call it the *modal background*. I also assume, following Kratzer, that the value of the modal base is generally a *consistent* set of propositions (i.e., R^f_w is non-empty for all f, w). The ordering source is not subject to this condition. It may be inconsistent, both internally and with the modal base.

The role of the ordering source in Kratzer's Ordering Semantics is to induce a binary relation on the set of possible worlds, often informally characterized as a relation of comparative distance from an "ideal" state of affairs (or a "stereotypical" one, depending on the modal flavor in question). At each world w, the order induced by g, here written $\leqslant^g_w$, is defined as follows: $v \leqslant^g_w u$ if and only if for all $p \in g(w)$, if $u \in p$ then $v \in p$. I write '$v <^g_w u$' for the statement that $v \leqslant^g_w u$ while not $u \leqslant^g_w v$, and '$\leqslant^g$' for the function mapping worlds to the corresponding orders under g. Note that $\leqslant^g_w$ is guaranteed to be reflexive and transitive — that is, a pre-order.

Conditional sentences are interpreted along essentially the same lines. The main idea, a generalization of Lewis's (1975) observation that '*if*'-clauses can serve as restrictors of quantifier domains, is that the '*if*'-clause does not introduce its own operator, but merely restricts the domain of a modal operator that is independently present in the structure. Thus for instance, the conditional in (2) is interpreted just like its non-conditional counterpart (1a), except that the relevant modal base is obtained from the contextually given f by adding to it the denotation of the antecedent — here, the proposition that the lights are on.

(2) If the lights are on, John must be at home. $must_{lights\ on}(John\ home)$

More precisely, for a modal base f and proposition p, I define the *update of f with p*, written $f[p]$, to be the function which maps every world w to the set $f(w) \cup \{p\}$. Clearly $f[p]$ is itself a conversational background, and the modal background $R_w^{f[p]}$ is a subset of R_w^f: it comprises just those worlds in R_w^f at which p is true. The conditional (2) then is true relative to f and g just in case its matrix clause (1a) is true relative to $f[lights\ on]$ and g.

1.2 Ordering frames

To facilitate the comparison between different versions of Ordering Semantics, I adopt from Lewis (1981) the notion of an *ordering frame*. (Lewis's definition differs from mine in some respects. See Section 3 below for details.)

Ordering frame An *ordering frame* is a triple $\langle W, R, O \rangle$, where W is a non-empty set (of possible worlds), R is a function from W to subsets of W, and O is a function from W to binary relations on W.

In Kratzer's semantics, each pairing of a modal base f with an ordering source g defined on W uniquely determines an ordering frame $\left\langle W, R^f, \leqslant^g \right\rangle$, where for all worlds w we define R_w^f to be $\cap f(w)$ (the modal background at w) and $\leqslant_w^g$ to be the pre-order induced by g at w, as defined above. I refer to this frame as the *Kratzer ordering frame*, or simply the *Kratzer frame* for f, g.

In the following, when I attribute certain properties to R, I generally mean to say that the values R_w at all worlds w have those properties. Specifically, I assume throughout that R is "non-empty", meaning R maps each world to a non-empty set. Like Kratzer (but unlike Lewis) I allow for the possibility that R_w does not contain w. Furthermore, in attributing properties to O, I mean to say that for all worlds w, the restriction of O_w to R_w has those properties. Thus for instance, I call O "reflexive" if the restriction of O_w to R_w is reflexive for all w. All Kratzer frames have reflexive and transitive orderings, but the properties of O will vary once we look beyond Kratzer frames. In most of this paper, however, I stick to Kratzer frames because they are the most familiar case, and I explicitly mark this restriction by using the symbols R^f and $\leqslant^g$ for the relevant parameters.

1.3 Two notions of necessity

Various definitions of necessity and possibility can be given relative to an ordering frame $\langle W, R^f, \leqslant^g \rangle$. Kratzer herself adopted the following from Lewis (1981).[1]

(KN) p is a *Kratzer necessity* at w relative to $R^f, \leqslant^g$ if and only if for all $u \in R^f_w$ there is some $v \in R^f_w$ such that (i) $v \leqslant^g_w u$ and (ii) for all $z \in R^f_w$, if $z \leqslant^g_w v$ then $z \in p$.

To understand this definition, it may be helpful to read (KN) in a "procedural" way, as an instruction for checking whether p is a Kratzer necessity, by examining paths through R^f_w which follow the links given by the order $\leqslant^g_w$ (that is, such that the next world is always at least as "good" as the last). Then p is a necessity if, starting from anywhere in R^f_w, it is always possible to reach a p-world from which only p-worlds are reachable. A corresponding strong notion of possibility can be defined as the dual of (KN).

Kratzer herself noted that (KN) is a cumbersome definition for a simple concept: Intuitively the idea is that p need not be true at all worlds in the modal background; rather, it is sufficient that p be true at those worlds in the modal background that are "closest" to the ideal dictated by g at w, that is, the worlds that are "best" according to the relevant criteria. However, the most straightforward formal implementation of this idea is problematic for technical reasons:

> As we can't assume that there have to be such things as *closest* worlds, the definition is rather complicated. It resembles the one David Lewis gives for counterfactuals.
>
> (Kratzer 1981: 48; emphasis in the original)

> The definition would be less complicated if we could quite generally assume the existence of such 'closest' worlds.
>
> (Kratzer 1991: 644)

Authors generally agree that (KN) is cumbersome. Kratzer's quote suggests that if we could count on the existence of closest worlds, then we could work with a simpler definition of necessity. Intuitively, instead of checking which worlds are reachable via various paths through the modal background, one could simply inspect the closest worlds and be done. Lewis (1973, 1981) had brought up this idea but simultaneously raised the worry that if we were to restrict ourselves to models in

1 Kratzer (2012) calls this notion "necessity". Kratzer (1981) calls it "human necessity", setting it apart from "simple necessity", which is defined in terms of universal quantification over the modal background.

which the existence of closest worlds was guaranteed, our theory would no longer do justice to the full richness of our modal discourse and the reasoning behind it. Therefore Lewis opted against the restriction. Kratzer followed him.

Others disagree, however, arguing that for the purposes of capturing the semantic properties of our modal and conditional language, not much is lost in coverage, and much is gained in simplicity, by assuming the existence of closest worlds (see Fn. 2 below for references). Specifically, the simpler definition that could be used in this case is spelled out in terms of the set of *minimal worlds* in the modal background under the order induced by the ordering source.

(LAN) p is an *LA necessity* at w relative to $R^f, \leqslant^g$ if and only if $\min(R^f_w, \leqslant^g_w) \subseteq p$.

The label "LA necessity" is shorthand for "Limit Assumption necessity", hinting at a special provision whose role is to ensure that (LAN) can safely be substituted for (KN).

The rest of this paper is dedicated to an investigation of the Limit Assumption. In Section 2 I list a variety of more or less precise ways in which it has been formulated in the literature, map them into a small set of distinct formal properties, and show that the most commonly encountered formulation — as the condition that the set of minimal worlds in the modal background be non-empty — does not serve its intended purpose on Kratzer frames. Section 3 traces this problem back to the early writings on the Limit Assumption by David Lewis. The formulation in terms of the existence of minimal worlds works for him; however, it does not have the analogous effect in Kratzer's framework due to subtle differences in the properties of the frames. The fact that later authors have frequently overlooked this difference is not due to any failure to read Lewis, but rather to an erroneous or at least misleading claim Lewis himself made about Kratzer frames. The upshot is that different frames call for different Limit Assumptions. Section 4 adds a further layer of complexity to this picture by showing that different interpretations of the modal operators also determine which version of the Limit Assumption is appropriate. Section 5 concludes the paper.

2 The Limit Assumption

We now have two notions of necessity, the cumbersome (KN) and the simpler (LAN). Those who prefer to work with the latter typically invoke the *Limit Assumption* to guard against unwelcome consequences of their choice. It is not my goal here to discuss the arguments for and against the simplification *per se*, nor do I take sides in that debate.[2] Rather, I am interested in a more fundamental question: What exactly *is* the Limit Assumption, and how should it be formulated in order for it to play its advertised role?

Let us start by noting what there is generally no disagreement about, namely the *purpose* of the Limit Assumption. Its purpose is to identify those circumstances under which it is safe to substitute one notion of necessity for the other without thereby changing the truth values of any sentences (specifically, of necessity statements) in the object language. I state this desideratum as the following *LA Postulate*:

(LAP) p is a Kratzer necessity at w relative to $R^f, \leqslant^g$ if and only if p is an LA necessity at w relative to $R^f, \leqslant^g$.

2.1 The status of (LAP): models vs. frames

What kind of statement is (LAP)? Following a line of inquiry familiar from modal logic, we may ask on which frames $\langle W, R^f, \leqslant^g \rangle$ it is *valid*, in the sense that its truth for all propositions p and worlds w is guaranteed by properties of R^f and $\leqslant^g$. This is the approach that many authors seem to adopt implicitly. But on some reflection it turns out that there is another, arguably more interesting way to look at it. Since this issue frames some of the discussion below, in this subsection I will set up some preliminary distinctions.

Taking some inspiration from the classic correspondence results of modal logic, we may state the Limit Assumption in terms of a property of ordering frames. Which property? Presumably the one which verifies a statement of the following form.

(Correspondence) (LAP) is valid on a frame $\langle W, R^f, \leqslant^g \rangle$ if and only if $\langle R^f, \leqslant^g \rangle$ has property $\mathscr{P}$.

In the discussion below, I will examine (LAP) from this perspective and show that there is a widespread misconception in the literature as to what the sought-after property $\mathscr{P}$ is. But before entering that discussion, it is worth noting that the applicability of both (Correspondence) and the earlier (LAP) is severely limited: While they do get at the question of when the two notions of necessity coincide for "bare" modal sentences like (1a), they do not begin to address the corresponding question for conditionals like (2).

2 Arguments for and against (LAN) were discussed by Herzberger (1979), Lewis (1981), Warmbrōd (1982), Pollock (1984), among others. The assumption that there are closest worlds is also an integral part of Stalnaker's (1968) theory of conditionals; there, in conjunction with the assumption that there is *at most* one closest world, it implies *Conditional Excluded Middle*; see Stalnaker (1981, 1984), Swanson (2011a) and references therein.

(1a) John must be at home. $must(John\ home)$

(2) If the lights are on, John must be at home. $must_{lights\ on}(John\ home)$

For recall that (2) is interpreted relative to a modal base f by evaluating (1a) relative to the updated modal base $f[lights\ on]$. More generally, a conditional '*if* p, *(must)* q' is true at w relative to R^f and $\leqslant^g$ just in case q is a necessity at w relative to $R^{f[p]}$ and $\leqslant^g$. So to ensure that the intersubstitutability of Kratzer necessity and LA necessity on a frame $\langle W, R^f, \leqslant^g \rangle$ extends to the interpretation of conditionals with antecedent p, (LAP) must hold of $R^{f[p]}$ and $\leqslant^g$. Now, this is a dramatic shift when it comes to the question of correspondence: once we have found the property $\mathscr{P}$ required to flesh out the above statement of (Correspondence), now the question becomes which property the *original* frame must have in order to ensure that the *derived* frame has property $\mathscr{P}$. And of course, the relevant constraint should ensure this for any proposition denoted by a conditional antecedent in the language, not just some particular proposition p. Thus we are looking for a property $\mathscr{Q}$ which validates a statement of the following form:

(Correspondence$^+$) All $\left\langle R^{f'}, \leqslant^g \right\rangle$ have property $\mathscr{P}$, where f' is either f or derivable from f via update with conditional antecedents, if and only if $\langle R^f, \leqslant^g \rangle$ has property $\mathscr{Q}$.

Now, what $\mathscr{Q}$ should be depends on which modal backgrounds f' are being quantified over. There are at least two ways to approach this latter question. The first is not to impose any constraints: any subset of R^f_w could in principle be the derived modal background $R^{f[\varphi]}_w$ for some antecedent φ, therefore $\mathscr{Q}$ should be the property (whatever it is) which ensures that all subsets of R^f_w have property $\mathscr{P}$, for all worlds w. This approach has the advantage of being in line with standard practice in correspondence theory. However, we will see below that from this perspective the question of what $\mathscr{Q}$ is becomes rather predictable and trivializes the original question what $\mathscr{P}$ was. So there is still some independent motivation for studying the latter question in its own right, if only because it is the one that most authors have been interested in in the literature.

The second way to approach the quantification over f' in (Correspondence$^+$) is to leave room for non-trivial constraints on which propositions can be the denotations of conditional antecedents. This approach is relevant in the present context because it was explicitly endorsed by David Lewis (see Section 3 below) and therefore underlies some of the most important writings on the Limit Assumption. Now, if there are to be constraints on which propositions can be denoted by conditional antecedents, then what property $\mathscr{Q}$ is depends on those constraints; but if those constraints are not explicitly defined, then $\mathscr{Q}$ cannot be determined. This is the situation in Lewis's theory. The LA Postulate is still well worth exploring in such a framework — as witnessed by Lewis's own writings — but only its manifestation as property $\mathscr{P}$. Assuming the Limit Assumption then amounts to imposing the left-hand side of (Correspondence^{+}) — that is, the condition that all derivable $\left\langle R^{f'}, \leqslant^{g} \right\rangle$ have property $\mathscr{P}$. But this is a constraint on *models*, not *frames*, since the set of derivable backgrounds is not determined by the frame alone, but also by the language and its interpretation.

The upshot of this discussion is that the question of property $\mathscr{P}$ is the more interesting one, and perhaps the only one that can be sensibly asked, depending on how exactly one reads the LA Postulate. Consequently, this paper is largely concerned with property $\mathscr{P}$. I will return to this discussion in Section 2.6, after some formal notions are clarified. To keep things simple, until further notice I limit the discussion to non-conditional necessity statements, i.e., ones without '*if*'-clauses. Those who prefer to consider conditionals the basic case, as Lewis did, are welcome to insert tautological antecedents.

2.2 Informal statements

The following brief and non-exhaustive list of quotations from the recent literature, with notation adjusted to the conventions of this paper, illustrates some of the diversity in the ways in which the Limit Assumption is stated.

- [T]here is a unique best set of worlds (Portner 1998)
- [T]he relation has minimal elements, ... there always are accessible worlds that ... are better than any world they can be compared with via $<^{g}_{w}$ (von Fintel 1999, von Fintel & Heim 2011)
- [T]here always exist closest worlds (Huitink 2005)
- For all $u \in R^{f}_{w}$ there exists $v \in \min(R^{f}_{w}, \leqslant^{g}_{w})$ such that $v \leqslant^{g}_{w} u$ (Schwager 2005)
- [F]or any world w and set of worlds X ... there is always at least one world w' in X that [is $\leqslant^{g}_{w}$-minimal] (Alonso-Ovalle 2008)[3]
- [T]he set of closest accessible worlds from a world will be non-empty and uniquely determined by a modal base and an ordering source (Nauze 2008)

3 Alonso-Ovalle states that at least one world w' in X "comes closest to w". This is what minimality amounts to in Kratzer's (and Lewis's) analysis of counterfactuals, which assumes *strong centering* — i.e., that $w <^{g}_{w} w'$ for all w, w'.

- [T]here always are accessible worlds that come closest to the ideal (Hacquard 2011)
- If R^f_w is non-empty, then so is $\min(R^f_w, \leqslant^g_w)$ (Kaufmann 2012)
- [T]here are $\leqslant^g_w$ best (or tied for best) worlds (Silk 2012)
- [E]very linearly ordered chain within the partial order terminates in a set of minimal worlds (Cariani, Kaufmann & Kaufmann 2013)
- [A]n inner domain always assigns a non-empty set to any possible world[4] (Knobe & Szabø 2013)
- [F]or all possible worlds w, for all $v \in R^f_w$ there is a $u \in \min(R^f_w, \leqslant^g_w)$ such that $u \leqslant^g_w v$ (Kaufmann & Kaufmann 2015)
- $\leqslant^g_w$ restricted to R^f_w is well-founded (Condoravdi & Lauer 2016)[5]

All of these statements are in some way or other about the existence of minimal worlds, but they vary on the question which set or sets must contain minimal worlds. Some are too underspecified to be sure exactly what condition is being expressed (this includes Huitink 2005, Silk 2012, Kratzer's above reference to the existence of closest worlds "quite generally", as well as Cariani et al. 2013).[6] Among the ones that do state clearly which sets must contain minimal words, most seem to say that it is the modal background. Below, I refer to this version as the *Singular Limit Assumption (SLA)* since for each $\langle f, g, w\rangle$ it requires of a single set of worlds that it contain minimal elements. According to others there must be minimal worlds in all (presumably non-empty) sets of worlds (Alonso-Ovalle 2008), or in all non-empty subsets of the modal background (Condoravdi & Lauer 2016). Since none of the definitions of necessity refers to worlds outside of the modal background, I take the version that quantifies over all non-empty subsets of the modal background as representative of this variant, which I dub the *Powerset Limit Assumption (PLA)*. Finally, we will see below that the version of Schwager (2005) and Kaufmann & Kaufmann (2015) differs from both (SLA) and (PLA).

Given the variety of non-equivalent versions of the Limit Assumption, it is noteworthy that there is little if any discussion of the question which one is correct.

4 Knobe and Szabø's "inner domain" is a function mapping worlds directly (without the mediation of a set of propositions) to *partial orders* on worlds (i.e., ones that are reflexive, transitive, and antisymmetric). Antisymmetry is not assumed by Kratzer, but this difference does not affect the issues I discuss in this paper.

5 $\leqslant^g_w$ is well-founded on R^f_w iff all non-empty subsets of R^f_w have $\leqslant^g_w$-minimal elements.

6 The intention behind the statement in Cariani et al. (2013) was to require that every maximal chain have a non-empty intersection with the set of minimal worlds. This is equivalent to the *Cutset Limit Assumption* (CLA) of Section 4 below.

Most authors gesture towards Lewis (1973, 1981) for details. I turn to this connection below (Section 3) and show that Lewis's writings are a source of confusion, rather than clarity, on the issue. First, however, I show that the two main versions defined so far, the Singular and the Powerset Limit Assumptions, are both wanting (the former in a more damning sense than the latter).

2.3 Singular LA

The first version of the Limit Assumption requires there to be minimal worlds in the modal background. In the following statement and in similar ones below, I assume universal quantification over the world variable w.

(SLA) $\min(R_w^f, \leqslant_w^g)$ is non-empty.

Do (KN) and (LAN) coincide whenever (SLA) holds? The answer is "no". On the one hand:

Fact 1
(SLA) is not sufficient for the validity of the LA Postulate.

Proof. The following is a counterexample.

(3) **Counterexample to (SLA) ⇒ (LAP)**
Let W be the set of natural numbers. Further, for all $w \in W$ let
a. $f(w) = \{W\}$;
b. $g(w) = \{\text{EVEN}\} \cup \{\{n, n+2, n+4, \ldots\} | n \in \text{ODD}\}$.

For instance, some of the propositions in $g(0)$ are listed in (4):

(4) $\{\text{EVEN}, \{1,3,5,7,9\ldots\}, \{3,5,7,9,\ldots\}, \{5,7,9,\ldots\}, \ldots\}$

So $R_0^f = W$, and the pre-order induced by $g(0)$ is as shown in Figure 1. Clearly the set of minimal worlds $\min(R_0^f, \leqslant_0^g)$ is non-empty: it is just the set of even numbers. Thus (SLA) is satisfied, and '*even*' is an LA necessity at 0. However, there is also an infinite descending chain of odd numbers; therefore '*even*' is not a Kratzer necessity at 0. □

Thus satisfaction of the Singular Limit Assumption does not guarantee that the substitution of LA necessity for Kratzer necessity is safe. On the other hand, while (SLA) is not *sufficient* for the validity of the LA Postulate, it is *necessary* for the latter.

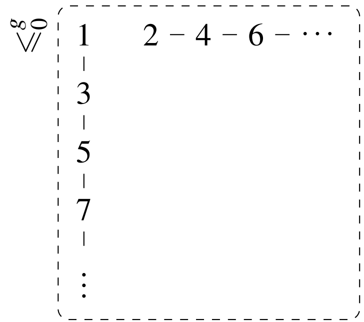

Figure 1 Induced pre-order for (3). The dashed box indicates R^f_0.

Fact 2
(SLA) is necessary for the validity of the LA Postulate.

Proof. Suppose for *reductio* that $\langle W, R^f, \leqslant^g \rangle$ (i) validates the LA Postulate but (ii) does not satisfy (SLA). By (ii), there is a world w for which $\min(R^f_w, \leqslant^g_w)$ is empty. Thus any proposition is (vacuously) an LA necessity at w, including both R^f_w and its complement $\overline{R^f_w}$. But since R^f_w is not empty, $\overline{R^f_w}$ is not a Kratzer necessity at w (regardless of $\leqslant^g_w$), contradicting (i). □

2.4 Powerset LA

The second version of the Limit Assumption imposes the stronger constraint that there be minimal worlds not only in the modal background itself, but also in all of its non-empty subsets. Generally this property is known as *well-foundedness* of $\leqslant^g_w$ (here restricted to R^f_w); cf. Footnote 5 above.

(PLA) For all $X \subseteq R^f_w$, if X is non-empty then so is $\min(X, \leqslant^g_w)$.

(PLA), like (SLA), fails to characterize the exact circumstances under which the LA Postulate is valid. However, while (SLA) was necessary but not sufficient, (PLA) is sufficient but not necessary.

Fact 3
(PLA) is sufficient for the validity of the LA Postulate.

Proof. I only show that given (PLA), (LAN) implies (KN) (the converse always holds). Suppose $\langle W, R^f, \leqslant^g \rangle$ satisfies (PLA). For arbitrary w and $u \in R^f_w$, let X_u be the set of worlds $v \in R^f_w$ such that $v \leqslant^g_w u$. By (PLA), $\min(X_u, \leqslant^g_w)$ is non-empty

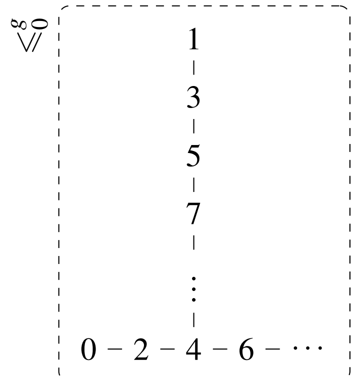

Figure 2 Induced pre-order for (5). The dashed box indicates R_0^f.

since X_u is a non-empty subset of R_w^f. Furthermore, since X_u includes all $v \in R_w^f$ such that $v \leqslant_w^g u$ (i.e., X_u is a *lower* subset of R_w^f — see Section 2.5 for definition), $\min(X_u, \leqslant_w^g)$ is a subset of $\min(R_w^f, \leqslant_w^g)$. Since u was chosen arbitrarily, (i) for every world u in R_w^f there is some $\leqslant_w^g$-minimal world $v \leqslant_w^g u$ in R_w^f; and, since v is $\leqslant_w^g$-minimal in R_w^f, (ii) any world $z \leqslant_w^g v$ in R_w^f is $\leqslant_w^g$-minimal in R_w^f. Let p be true at all $\leqslant_w^g$-minimal worlds in R_w^f, thus p is an LA necessity at w. Then by (i) and (ii), p is also a Kratzer necessity at w. □

On the other hand, the LA Postulate may be valid even in case (PLA) is not satisfied.

Fact 4
(PLA) is not necessary for the validity of the LA Postulate.

Proof. Here is a counterexample.

(5) **Counterexample to (LAP) ⇒ (PLA)**
Let W and f be as before, and for all $w \in W$, let
a. $g(w) = \{\text{EVEN} \cup \{n, n+2, n+4, \ldots | n \in \text{ODD}\}\}$.

Some of the propositions in $g(0)$ are listed in (6). The resulting order $\leqslant_0^g$ is shown in Figure 2.

(6) $\{\{0,1,2,3,4,5,6,\ldots\},\{0,2,3,4,5,6,\ldots\},\{0,2,4,5,6,\ldots\},\ldots\}$

The order $\leqslant_0^g$ is not well-founded on R_0^f: for instance, ODD is a non-empty subset of R_0^f, but $\min(\text{ODD}, \leqslant_0^g)$ is empty. Still, any LA necessity at 0 is also a Kratzer necessity at 0. To see this, just note that p is an LA necessity at 0 iff $\text{EVEN} \subseteq p$, and that all odd numbers are strictly bettered by the even numbers. Since by assumption R^f_w and $\leqslant^g_w$ are the same for all w in W, (LAP) is valid for f, g. □

Although (PLA), like (SLA), fails to characterize the class of frames on which the LA Postulate is valid, in the case of (PLA) the failure is less problematic for practical purposes: if one's primary concern is to ensure that the LA postulate holds, sufficiency is key. In this sense, in linguistic practice (PLA) is a "safer" constraint to impose than (SLA). But since neither *corresponds to* the LA Postulate in the technical sense, it is still worth asking which property does.

2.5 Lower set LA

In the logical space between (SLA) and (PLA) there is room for at least two other versions, one of which gives us the right answer to the question at hand and the other of which will serve us well further below. The first of the two can be formulated in several equivalent ways; one is in terms of quantification over non-empty *lower subsets* — equivalently, *downward closed* subsets — of R^f_w under $\leqslant^g_w$.

Lower subset A subset X of $\left\langle R^f_w, \leqslant^g_w \right\rangle$ is a lower subset iff for all $u \in X$ and $v \in R^f_w$, if $v \leqslant^g_w u$ then $v \in X$.

(LLA) For all lower subsets X of R^f_w, if X is non-empty then so is $\min(X, \leqslant^g_w)$.

The difference between (LLA) and (PLA) is subtle but crucial: precisely the kind of counterexample presented in (5) above is ruled out by (LLA). In (5), even as the LA Postulate was valid, (PLA) was violated because the set of odd numbers did not contain a minimal element. But the set of odd numbers is not a lower subset in that frame, and the frame does satisfy (LLA).

More generally, (LLA) corresponds to the LA Postulate.

Fact 5
(LLA) is sufficient and necessary for the validity of the LA Postulate.

Proof. (1) Sufficiency. I only show that given (LLA), (LAN) implies (KN). Suppose $\left\langle W, R^f, \leqslant^g \right\rangle$ satisfies (LLA) and let w be an arbitrary world in W at which p is an LA necessity. For an arbitrary world $u \in R^f_w$, let X_u be the set of all worlds $x \in R^f_w$ such that $x \leqslant^g_w u$. Since X_u is a lower subset in $\left\langle R^f_w, \leqslant^g_w \right\rangle$, $\min(X_u, \leqslant^g_w)$ is a subset of $\min(R^f_w, \leqslant^g_w)$ and non-empty by (LLA). Let $v \in \min(X_u, \leqslant^g_w)$. By assumption, p is true at all worlds in $\min(R^f_w, \leqslant^g_w)$, thus *a forteriori* at all worlds z in R^f_w such that $z \leqslant^g_w v$. Hence p is a Kratzer necessity at w.

(2) Necessity. Suppose for reductio that $\langle W, R^f, \leqslant^g \rangle$ (i) validates the LA Postulate but (ii) does not satisfy (LLA). By (ii) there is a world $w \in W$ such that for some non-empty lower subset X of R^f_w, $\min(X, \leqslant^g_w)$ is empty. Now, $\min(R^f_w, \leqslant^g_w)$ is either empty or not; in either case, it is a subset of $\overline{X}$, hence $\overline{X}$ is an LA necessity at w. However, since X is a non-empty lower subset, $\overline{X}$ is not a Kratzer necessity at w. Hence f, g do not validate the LA Postulate, contradicting (i). □

Before moving on, I mention two alternative formulations that identify the same class of frames. First a definition.

Maximal antichain X is an *antichain* in $\left\langle R^f_w, \leqslant^g_w \right\rangle$ iff X is a subset of R^f_w and all worlds in X are pairwise incomparable to each other under $\leqslant^g_w$. X is a *maximal* antichain iff X is an antichain and every world in the relative complement $R^f_w - X$ is comparable under $\leqslant^g_w$ to some world in X.

Note that whenever R^f_w is non-empty, so are its maximal antichains.

Now on to the equivalent conditions on frames. One of them, here dubbed (KLA), is due to Schwager (2005) and Kaufmann & Kaufmann (2015). It lends itself well to an intuitive paraphrase in terms of paths through the modal background, which I mentioned in Section 1.3 above: it states that from every world in the modal background there is a path to a minimal world; or equivalently, that every non-maximal chain can be extended to a maximal chain with a least element. The second one, dubbed (ALA) for "Antichain LA", will be relevant below.

(KLA) For all u in R^f_w, there is some v in $\min(R^f_w, \leqslant^g_w)$ such that $v \leqslant^g_w u$.[7]

(ALA) Some subset of $\min(R^f_w, \leqslant^g_w)$ is a maximal antichain in $\left\langle R^f_w, \leqslant^g_w \right\rangle$.

Fact 6
(LLA), (KLA), and (ALA) are valid on the same class of frames.

Proof. (LLA) $\Rightarrow$ (KLA). Suppose $\langle W, R^f, \leqslant^g \rangle$ satisfies (LLA). For arbitrary $w \in W$ and $u \in R^f_w$, let X_u be the set of worlds $v \in R^f_w$ such that $v \leqslant^g_w u$. Since X_u is a non-empty lower subset of $\left\langle R^f_w, \leqslant^g_w \right\rangle$, $\min(X_u, \leqslant^g_w)$ is non-empty by (LLA); furthermore, $\min(X_u, \leqslant^g_w)$ is a subset of $\min(R^f_w, \leqslant^g_w)$. Now, either $u \in \min(X_u, \leqslant^g_w) \subseteq \min(R^f_w, \leqslant^g_w)$, or there is some $v \in \min(X_u, \leqslant^g_w) \subseteq \min(R^f_w, \leqslant^g_w)$ such that $v \leqslant^g_w u$.

7 The statement in this form is only applicable because the ordering relation is reflexive in Kratzer's semantics. Applied to irreflexive orders $<^g_w$ (as in Lewis 1981), it should read "…such that $v <^g_w u$ or $v = u$".

(KLA) $\Rightarrow$ (ALA). In fact, (KLA) implies that every maximal antichain in $\min(R^f_w, \leqslant^g_w)$ is a maximal antichain in $\left\langle R^f_w, \leqslant^g_w \right\rangle$. To see this, let C be a maximal antichain in $\min(R^f_w, \leqslant^g_w)$[8] and note that by (KLA) and the properties of $\leqslant^g_w$, every world in R^f_w is comparable to some element of C.

(ALA) $\Rightarrow$ (LLA). Suppose $\left\langle W, R^f, \leqslant^g \right\rangle$ satisfies (ALA). For $w \in W$, let X be an arbitrary non-empty lower subset of R^f_w, and u be an arbitrary world in X. Either $u \in \min(R^f_w, \leqslant^g_w)$, or by (ALA) and the fact that X is a lower set, there is some $v \in \min(R^f_w, \leqslant^g_w)$ such that $v \leqslant^g_w u$. $\square$

2.6 A note on quantification over subsets

Before we proceed, let me sort out a potential source of confusion about the quantification over sets of worlds other than the modal background in formulating the Limit Assumption, as was done for (LLA) and (PLA) above. This issue links back to the informal distinction between properties $\mathscr{P}$ and $\mathscr{Q}$ in Section 2.1 above.

There are two potential reasons for quantifying over subsets of the modal background. The first is what we saw in this section: for the validity of the Limit Assumption on a given modal background, it is not enough that it contain minimal worlds; rather, all of its lower subsets must do so. Notice that this argument did not rely on the interpretation of conditionals. It would hold just the same for a language containing only bare modals, or for the conditional-free fragment of English. In terms of the earlier discussion, (LLA) is property $\mathscr{P}$.

The second reason for quantifying over subsets of the modal background involves conditionals. Since they are interpreted relative to a derived modal background obtained via update with the antecedent, (LLA) must be guaranteed to hold of that derived modal background. More generally, call a modal background "reachable" from R^f just in case it either equals R^f or can be derived from R^f in a finite sequence of updates with arbitrary antecedents. (Derivability by a single update would not be general enough, given Kratzer's (2012: 105) treatment of stacked '*if*'-clauses in terms of sequences of updates.) Then (LLA) would have to hold of all reachable modal backgrounds. In Section 2.1 I dubbed the property of R^f that ensures this $\mathscr{Q}$.

Although these two reasons for quantifying over subsets are conceptually distinct, they are not logically independent. For instance, consider the interpretation of sentences relative to R^f_w and $\leqslant^g_w$ at an arbitrary world w, and suppose every subset of R^f_w is reachable. Then property $\mathscr{P}$ must hold of all subsets of the modal background. What $\mathscr{P}$? It turns out that the distinctions discussed above no longer matter in this

8 To see that such a set C exists, note that $\min(R^f_w \leqslant^g_w)$ is itself pre-ordered and hence susceptible to Swanson's (2011b) adaptation of Kurepa's (1953) antichain principle.

case: each of (SLA), (LLA) and (PLA) holds of all subsets of R_w^f iff (PLA) holds of R_w^f. So we could pick any one of the three as our property $\mathscr{P}$ (and it would follow automatically that (PLA) is property $\mathscr{Q}$).

But while any of these three choices would have validate the LA Postulate for the particular case at hand, only (LLA) would generalize. For instance, (SLA) would be too weak and (PLA) would be overly strong for a language which does not contain conditionals, or for an interpretation under which not all subsets of the modal background are reachable. In other words, the correctness of (SLA) and (PLA) depends on the object language and its interpretation, in a way in which the correctness of (LLA) does not.

In this connection, I would like to briefly return to a point I alluded to in Section 2, namely that the question of which property $\mathscr{Q}$ ensures that $\mathscr{P}$ holds of all reachable backgrounds is uninteresting when viewed as a matter of "correspondence" in the sense familiar from modal logic. I am now in a better position to explain why this is so.

Standard correspondence results relate frame properties to the validity of certain forms of object-language statements, where validity on a frame means truth in all models defined on that frame. If we are dealing with a language that does not contain conditionals, the LA Postulate corresponds to (LLA) in this sense. However, conditionals shift the modal background, what they shift it to depends on the interpretation of the antecedent, and that interpretation is not determined by the frame but by the model. Unless the class of models is restricted in some non-trivial way, *any* set of worlds *could* be the denotation of a given antecedent in *some* model, hence any subset of the derived modal background is reachable in principle. This leads again to the collapse of the distinctions between (SLA), (LLA) and (PLA) just discussed: each holds of all subsets iff the others hold of all subsets. Thus the differences between the three properties of modal backgrounds vanish from our view. I conclude that correspondence theory is too blunt an instrument to draw the distinctions I am interested in.

But truth in all possible models on a frame is generally not the issue in discussions of the Limit Assumption. Lewis addressed this matter directly in the first writings on the topic. I discuss Lewis's view in more detail in Section 3; for now, I just note that he imposed (SLA) on all reachable modal backgrounds while explicitly leaving room for subsets that violate (SLA): as long as they were not denoted by any antecedents, they did not invalidate the Limit Assumption as he saw it. I adopt this perspective, not least because it distinguishes between the various properties of modal backgrounds discussed in this section. Thus I treat the LA Postulate as a constraint on models, not on frames. My calling (LAP) a "postulate" (not an "axiom") is intended to highlight this choice.

3 Connection to Lewis

Kratzer's framework for the analysis of modality is similar to Lewis's semantics for counterfactuals in most important respects. Lewis's (1981) well-known comparison of the two showed them to be largely equivalent, safe for certain details which he argued to be immaterial for the resulting semantic theory. For my purposes, those details actually turn out to be significant. The discussion in this section focuses on the main differences; for a more general comparison of the frameworks, the reader is referred to Lewis (1981).

The crucial difference concerns the properties of the ordering relation. Lewis imposed a condition of *Comparability* which is lacking in Kratzer's version. In the slightly different but interconvertible versions of 1973 and 1981, Lewis implemented Comparability as *connectedness* in the former and as *almost-connectedness* in the latter. In (7) I list the properties assumed in the three approaches at issue here, along with commonly used labels for relations with the respective properties.[9]

(7) **Properties of $\mathbf{O_w}$**

a. Lewis (1973): $\leqslant_w$ reflexive, transitive, and connected *[strict weak order / total pre-order]*

b. Lewis (1981): $\angle_w$ irreflexive, transitive, and almost-connected *[weak order]*

c. Kratzer (1981, 1991): $\leqslant^g_w$ reflexive and transitive *[pre-order]*

Following Lewis (1981), I write '$u \leqq_w v$' to mean '$u \angle_w v$ or $u = v$'. Comparability (i.e., connectedness or almost-connectedness) allows Lewis to use a definition of necessity which differs from Kratzer's (LAN). I state it as (LN) below; note, though, that (LN) differs from Lewis's original version, which was only defined for conditionals. The difference is not profound for my purposes since Kratzer's treatment of non-conditionalized modals can be simulated by substituting conditionals with tautological antecedents.[10] The modification is intended to facilitate the comparison.[11]

9 All free variables in the following are universally quantified over. Reflexive: $u\ O_w\ u$. Transitive: if $u\ O_w\ v$ and $v\ O_w\ z$, then $u\ O_w\ z$. Connected: $u\ O_w\ v$ or $v\ O_w\ u$. Almost-connected: If $u\ O_w\ v$, then either $u\ O_w\ z$ or $z\ O_w\ v$ (or both). In place of almost-connectedness, Lewis (1981) imposes the equivalent condition that the complement of the symmetric closure of O_w be transitive.

10 The two do differ in their analysis of embedded conditionals, for instance of the form '*if p, then (if q, r)*'. On Kratzer's account this is equivalent to '*if p and q, then r*' (Kratzer 2012: p. 105), whereas Lewis would presumably analyze it as $p \mathrel{\Box\!\!\rightarrow} (q \mathrel{\Box\!\!\rightarrow} r)$. But none of this affects my point about the parallels in interpretation between Kratzer's unary modal operator and Lewis's binary one.

11 Lewis (1973) used the following version of necessity, which is suitable for the reflexive and connected relations he assumed there:

(LN) p is a *Lewis necessity* at w iff there is some $v \in R_w$ such that for all $z \in R_w$, if not $v \angle_w z$ then $z \in p$.

Lewis's first statement of the Limit Assumption appeared in his book on counterfactuals:

> The assumption that, for every world w and antecedent A that is entertainable at w, there is a smallest A-permitting sphere, I call the *Limit Assumption*. It is the assumption that as we take smaller and smaller antecedent-permitting spheres, containing antecedent-worlds closer and closer to w, we eventually reach a limit: the *smallest* antecedent-permitting sphere, and in it the *closest* antecedent-worlds. (Lewis 1973: 19-20 – emphasis in the original)

An antecedent is "entertainable" at w if it is true at some world in R_w. The smaller or larger "spheres" are unions of equivalence classes under the similarity order, which in Lewis's fanciful "Ptolomaic astronomy" (1973: 16) are pictured as onion-like concentric rings around the world of evaluation.

The above statement quantifies universally over conditional antecedents. This is where the discussion of the motivation for such quantification in Section 2.6 becomes relevant. If all subsets of R_w were possible antecedent denotations, then Lewis would be endorsing (if only indirectly) the Powerset Limit Assumption (PLA), the condition that all non-empty subsets of R_w contain minimal worlds. But Lewis himself took pains to emphasize that this was not his intention. In the paragraph preceding the quoted passage, he explicitly allowed for the possibility that not all subsets of R_w are relevant:

> If there are sequences of smaller and smaller spheres without end, then there are sets of spheres with no smallest member: take the set of all spheres in any such sequence. Yet it might still happen that for every entertainable antecedent in our language, there is a smallest antecedent-permitting sphere. For our language may be limited in expressive power so that not just any set of worlds is the set of A-worlds for some sentence A; and, in that case, it may never happen that the set of A-permitting spheres is one of the sets that lacks a smallest member, for any antecedent A. (Lewis 1973: 19)

(LN$_{1973}$) p is a *Lewis necessity* at w iff there is some $v \in R_w$ such that for all $z \in R_w$, if $z \leqslant_w v$ then $z \in p$.

This version highlights the parallelism with (KN): the latter merely involves an additional layer of universal quantification. Nevertheless, I largely focus on the 1981 version because there Lewis spelled out his comparison with Kratzer's semantics.

Thus in quantifying over all entertainable antecedents, Lewis did not intend to quantify over all subsets of R_w. And since he did not suggest that the denotations of antecedents can be read off the structure of the frame, we cannot draw any conclusions as to *which* subsets of R_w (other than R_w itself) his Limit Assumption requires to have minimal elements. I conclude that Lewis is best read as imposing (SLA) on all reachable modal backgrounds, and importantly, that he did not assume that the correctness of this move depended in any way on which modal backgrounds were reachable.

The second version of Lewis's Limit Assumption (slightly adjusted for notation) reads as follows:

> Unless no A-world belongs to R_w, there is some closest A-world to w.
> (Lewis 1981: 228)

Lewis (1981) said nothing on the question of which propositions qualify as antecedents. I take the absence of any such mention to indicate that his views had not changed since 1973. But this is problematic, for it leads me to conclude that he made a mistake in one of his proofs.

Lewis argued that in Kratzer models, Kratzer necessity reduces to LA necessity given the Limit Assumption alone, absent Comparability. But we already saw that that is not the case: (3) above is a counterexample. The source of the confusion can be pinpointed to the following passage from Lewis's proof. Here 'A' stands for the conditional antecedent.

> Given $h \in A \cap R_w$, let B be the set of all worlds $g \in A$ such that $g \leqq_w h$. By the Limit Assumption, since $h \in B \cap R_w$, we have j which is a closest B-world to w. (p. 231 – notation adjusted)

Note the absence of any suggestion to the effect that B is the denotation of an antecedent. Rather, as this quote shows, Lewis assumed that there are minimal worlds in arbitrary *principal ideals* of $A \cap R_w$.[12] But without Comparability, (SLA) does not warrant that assumption; (LLA) does.

I see only one way to avoid the conclusion that Lewis committed an error in his proof, and that is to assume that he had tacitly given up the possibility that the quantification over entertainable antecedents was in any way restricted. But this

12 While Lewis's statement only restricts B to worlds in A, his assumption of *strict centering* (that w is the least element under $\leqq_w$ and an element of R_w) ensures that B is a subset of $A \cap R_w$. An ideal in $\langle R_w, \leqq_w \rangle$ is a subset X of R_w that is (i) *directed:* for all $u, v \in X$ there is a $z \in X$ such that $u \leqq_w z$ and $v \leqq_w z$; and (ii) downward closed: for all $u, v \in R_w$, if $u \in X$ and $v \leqq_w u$ then $v \in X$. The principal ideal generated by $x \in R_w$ is the smallest ideal containing x. Thus in Lewis's proof, B is the principal ideal generated by h.

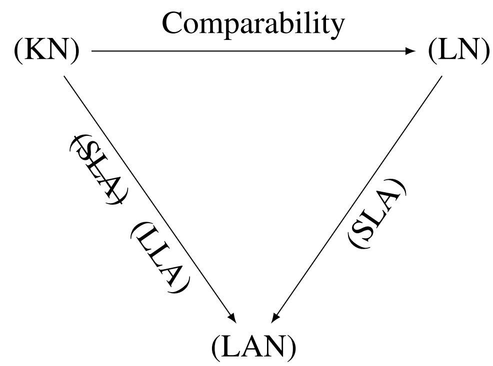

Figure 3 Three notions of necessity and the order properties that collapse them.

would mean that in effect his Limit Assumption now amounted to (PLA), which if true would be obscured by his own definition, and which moreover I find doubtful in view of his realism about possible worlds.

Before concluding this section, let me put the issue just discussed in the context of the other observations in Lewis (1981). Lewis correctly showed that Kratzer necessity reduces to Lewis necessity given Comparability, and that Lewis necessity further reduces to LA necessity given (SLA). Since Lewis himself generally assumes Comparability, Kratzer necessity and Lewis necessity never come apart for him, so his account is immune to the problem discussed in this subsection. (Another way to see this is to note that given Comparability, (LLA) reduces to (SLA).) It is only in the absence of Comparability, and thus only on Kratzer frames, that the two notions of necessity can come apart and (SLA) alone is insufficient. Figure 3 shows the relationships between the three notions of necessity discussed so far.

4 Necessity dependence

One upshot from the discussion so far is that there is no single "correct" Limit Assumption. Rather, which version delivers the desired result depends on the properties of the underlying ordering relation. Throughout the paper so far I did not question the correctness of (LLA), which relates Kratzer necessity to LA necessity.

But it is not universally agreed that Kratzer necessity is the right notion to capture our intuitions about the interpretation of modal and conditional sentences. Its correctness was recently questioned by Swanson (2011b), who argued that it makes counterintuitive predictions in certain cases, and offered an alternative version of necessity which, for those examples, is more in line with intuitions. This proposal is worth discussing in the present context because it has implications for the formulation of the Limit Assumption. Briefly put, if we follow Swanson's suggestion and adopt

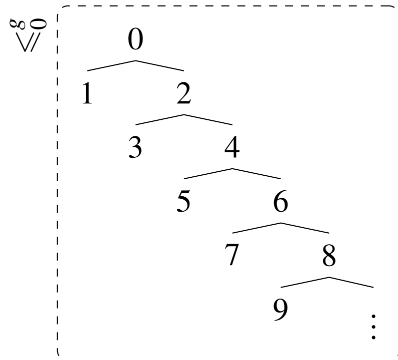

Figure 4 Induced pre-order for Example 1. The dashed box indicates R_0^f.

a new definition of necessity, then the job of the Limit Assumption is to validate a different LA Postulate, therefore it characterizes a different property of the ordering relation.

The following is an abstract version of Swanson's "Cheaper by the dozen" scenario.

Example 1. *As before, the set W of worlds is the set of natural numbers, with $f(0) = \{W\}$. The ordering source at 0 is the set*

$$\bigcup_{n \in \text{EVEN}} \{\{m | m \leq n, m \in \text{EVEN}\}, \{m | m \leq n, m \in \text{EVEN}\} \cup \{n+1\}\}$$

Some of the propositions in $g(0)$ are listed in (8). The resulting order is depicted in Figure 4.

(8) $\{\{0\}, \{0,1\}, \{0,2\}, \{0,2,3\} \{0,2,4\}, \{0,2,4,5\}, \{0,2,4,6\}, \ldots\}$

To give some content to this abstract picture, consider the romantic life of some person, Kim. Let the even numbers keep track of Kim's premarital relationships: for instance, in world 8 Kim has five lovers, numbered 0, 2, 4, 6 and 8. All premarital affairs n (for any even-numbered n) are of finite duration and have one of three possible outcomes: Either Kim remains alone (this is the case at world n); or Kim marries lover n (at world $n+1$), or Kim finds a new lover (at world $n+2$). Marriage and new love are both better than staying alone (both $n+1$ and $n+2$ are strictly better than n), but otherwise Kim tends towards contentment: no two relationships are comparable with each other. (Thus Kim's apparent preference for more rather than fewer premarital relationships is solely due to the fact that none of them lasts.)

Swanson shows that in a setting with this structure, some statements of necessity and possibility receive counterintuitive truth values when given an interpretation in terms of Kratzer necessity. For instance, it is easy to check that in the above example, Kratzer necessity would render (9a) true and (9b) false:

(9) a. Kim must get married.
b. Kim need not get married.

Swanson argues that both of these predictions are rather counterintuitive (cf. his (12) and (14)). Whether the argument is convincing is not my primary concern here (although I think Swanson has a point). Rather, what I want to show is that if we are moved by considerations of this sort to adopt a new version of necessity, as Swanson urges us to, then the Limit Assumption has to change as well.

Swanson, in presenting his fix for the theory, starts by recasting the familiar definition of Kratzer necessity relative to a frame $\langle R_w, \leqslant_w \rangle$ from a different perspective:

(ACN) p is an *antichain necessity* at w if and only if there is a maximal antichain B in $\langle R_w, \leqslant_w \rangle$ such that for all h in B and j in R_w, if $j \leqslant_w h$ then $j \in p$.

Swanson shows that (KN) and (ACN) are equivalent. The point of introducing the latter is that it facilitates the comparison with his proposed alternative definition.

Cutset $X \subseteq R_w$ is a cutset in $\langle R_w, \leqslant_w \rangle$ iff it has a non-empty intersection with every maximal chain in $\langle R_w, \leqslant_w \rangle$.

(SN) p is a *Swanson necessity* at w if and only if there is a cutset B in $\langle R_w, \leqslant_w \rangle$ such that for all h in B and j in R_w, if $j \leqslant_w h$ then $j \in p$.

As can be seen, the two definitions of necessity are very similar, safe for the difference between antichains and cutsets. In the above example, this difference is crucial. Specifically, the set ODD of odd numbers is a maximal antichain in the indicated pre-order, consequently '*odd*' is a Kratzer necessity. However, ODD is not a cutset, since it does not contain any elements of EVEN, the set of all even numbers, which is a maximal chain. Any cutset in this frame must contain an even number. Therefore '*odd*' is not a Swanson necessity. See Swanson (2011b) for more discussion.

Now, staying with the above example, notice that the set of minimal worlds is non-empty: it is just the set ODD of odd numbers. Therefore '*odd*' is an LA necessity. Moreover, every lower subset in the order contains one or more odd numbers as minimal elements. Thus both (SLA) and (LLA) hold. This is as it should be, given that '*odd*' is also a Kratzer necessity.

But while '*odd*' is both an LA necessity and a Kratzer necessity, it is not a Swanson necessity, even as both (SLA) and (LLA) hold. Thus neither of these limit assumptions characterizes the class of frames on which (SN) and (LAN) coincide.[13] What constraint *does* characterize this set of frames?

The answer is by now not surprising. I noted above that the Lower set Limit Assumption is equivalent to the Antichain Limit Assumption repeated here:

(ALA) Some subset of $\min(R_w, \leqslant_w)$ is a maximal antichain in $\langle R_w, \leqslant_w \rangle$.

The move from (KN) to (SN) would require a concomitant move from the Antichain Limit Assumption to the following *Cutset Limit Assumption*:

(CLA) Some subset of $\min(R_w, \leqslant_w)$ is a cutset in $\langle R_w, \leqslant_w \rangle$.

There are again alternative ways to state the same constraints, as before (recall from Fact 6 above that (ALA) is equivalent to (LLA) and (KLA)). For present purposes the following might be illuminating, given here without proof: (ALA) means that every non-maximal chain can be extended to a maximal chain which intersects $\min(R_w, \leqslant_w)$, whereas (CLA) means that every maximal chain intersects $\min(R_w, \leqslant_w)$.

(ALA) and (CLA) are not equivalent. In our example, the set of minimal worlds contains a maximal antichain (in fact, it *is* one) but not a cutset. So (ALA) is satisfied and (CLA) is not. Correspondingly, (KN) does and (SN) does not coincide with (LAN). More generally:

Fact 7
(CLA) is sufficient and necessary for the equivalence of (SN) and (LAN).

Proof. (1) Sufficiency. I only show that given (CLA), (LAN) implies (SN). Suppose $\langle R_w, \leqslant_w \rangle$ satisfy (CLA). For an arbitrary world w, suppose some proposition p is (i) a LA necessity at w but (ii) not a Swanson necessity at w. By (ii), there is no cutset $B \subseteq \langle R_w, \leqslant_w \rangle$ such that p is true at every world that is at least as good as some B-world. But by (i) and the assumption that (CLA) holds, there is such a cutset. Contradiction.

(2) Necessity. Suppose for reductio that (i) (SN) and (LAN) coincide on $\langle R_w, \leqslant_w \rangle$, which however (ii) does not satisfy (CLA). Now, $\min(R_w, \leqslant_w)$ may empty or not; in either case, by (ii), $\min(R_w, \leqslant_w)$ does not contain a cutset in $\langle R_w, \leqslant_w \rangle$. Thus there is at least one maximal chain in $\langle R_w, \leqslant_w \rangle$ which has an empty intersection

13 I note only in passing that (PLA), the third version of the Limit Assumption that I discussed above, also does not fit the bill. The above example does not show this because (PLA) does not hold. In fact, as in the earlier discussion, (PLA) is sufficient but not necessary for the equivalence of (SN) and (LAN). I omit the proof of sufficiency; for a counterexample to necessity, see Example (5) above. There, Kratzer necessity and Swanson necessity coincide.

$$\begin{tikzcd}
\text{(SN)} \arrow[r, "\text{SN/KN}"] \arrow[dr, "\text{(CLA)}"'] & \text{(KN)} \arrow[r, "\text{Comparability}"] \arrow[d, "\text{(LLA)} \equiv \text{(ALA)}"] & \text{(LN)} \arrow[dl, "\text{(SLA)}"] \\
& \text{(LAN)} &
\end{tikzcd}$$

Figure 5 Four notions of necessity and the order properties that collapse them.

with $\min(R_w, \leqslant_w)$. Let X be the union of all maximal chains with this property. Then $\overline{X}$ is an LA necessity but not a Swanson necessity at w, hence (SN) and (LAN) do not coincide, contradicting (i). □

Figure 5 is an extension of Figure 3, showing how Swanson necessity fits into the overall picture. The Cutset Limit Assumption reduces Swanson necessity to LA necessity, similarly to what the Lower set Limit Assumption and the Singular Limit Assumption do to Kratzer necessity and Lewis necessity, respectively.

The horizontal arrow leading from Swanson to Kratzer necessity indicates the order property under which the two coincide. The label "SN/KN" is but a placeholder for the correct statement of the sufficient and necessary conditions for their collapse, which I am at present not able to provide. I leave that question for another occasion.[14]

5 Conclusions

There is no such thing as "the" Limit Assumption. Once the goal is set — generally the goal is to ensure that the operative notion of necessity coincides with universal quantification over the set of minimal worlds — it becomes evident that what condition is sufficient and necessary to serve this purpose depends on properties of the ordering relation as well as the notion of necessity in question.

14 I conjecture that the sought-after condition is to be found somewhere in the neighborhood of the statement that every maximal antichain is a cutset, or that no maximal chain "stays above" any set of worlds (where X stays above Y iff for all $x \in X$ there is some $y \in Y$ such that $y < x$). See also Swanson 2011b, especially Thm. 3.

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