

A “de-Fregean” semantics (and neo-Gricean pragmatics) for modified and unmodified numerals*

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Abstract A challenge for the semantic and pragmatic analysis of modified numerals is how to account for ignorance implications about exact quantity. Superlative-modified numerals (*at least/most six*) systematically give rise to such implications while their comparative-modified counterparts (*more/fewer than six*) do not, but the distribution of ignorance implications with superlative modifiers is sensitive to how the numeral interacts with modals and other operators. In this paper, I demonstrate that a “de-Fregean” semantic analysis of modified and unmodified numerals as second-order properties of degrees that differ only in the kind of ordering relation they introduce supports a neo-Gricean pragmatic account of ignorance implications as Quantity implicatures, and derives the pattern of interaction with modals as a scopal phenomenon.

Keywords: Numerals, implicature, ignorance, free choice, degree quantification

1 Introduction

It is standard procedure on commercial airline flights to provide information about the safety features of the airline. As part of this demonstration, a flight attendant might utter a sentence like (1).

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- (1) This airplane has six emergency exits.

A passenger on the plane who is paying attention to the safety demonstration will take the flight attendant to be communicating the information that the airplane in question has exactly six emergency exits. If the passenger later discovers that the airplane has five or seven emergency exits, she will retroactively judge the flight attendant to have been mistaken, which may influence her attitude towards future flights on the airline; but she would have no reason based on the flight attendant's utterance alone to question his knowledge of the airplane's safety features.

If the flight attendant not only wants to provide information about emergency exits, but also wants to say something about how the plane compares to others of a similar type, he might instead make one of the utterances in (2), possibly prefaced by something to the effect of *Unlike other planes of similar design...*

- (2) a. This airplane has more than six emergency exits.
b. This airplane has fewer than six emergency exits.

The attentive passenger might wonder why the flight attendant is providing this extra information, but she would have no reason to doubt the quality of his knowledge about the airplane.

If, however, the flight attendant were to begin the safety demonstration with one of the following utterances, the attentive passenger would most likely look up in alarm:

- (3) a. This airplane has at least six emergency exits.
b. This airplane has at most six emergency exits.

The reason for the passenger's alarm is that in uttering (3a) or (3b), the flight attendant signals that he does not know what the actual number of emergency exits is; he merely knows the lower or upper bound. This in turn raises questions about his overall knowledge of the safety features of the airplane, and the passenger would not be unjustified in demanding to be let off the plane.

The challenge for semantic and pragmatic theory is to explain why the superlative modifiers in (3) imply ignorance about actual quantity, while the comparative modifiers in (2) do not, given that both sets of modifiers appear on the surface to be equally uninformative about actual number. Geurts & Nouwen 2007, in one of the first comprehensive analyses of this

phenomenon, effectively build this difference into the semantics of the two kinds of modifiers, arguing that superlative modifiers have an epistemic component that comparative modifiers lack. Subsequent research has taken a different strategy, aiming to derive the difference from the interaction of the meanings of the two classes of modifiers and independent semantic mechanisms (Nouwen 2010) or pragmatic principles (Büring 2008, Cummins & Katsos 2010, Coppock & Brochhagen 2013, Schwarz 2013, Rett 2014).

One argument in favor of the latter type of approach is that the absence vs. presence of ignorance inferences is not a feature specific to *more/fewer than* vs. *at least/most*, but instead supports a quite general classification of modifiers into two groups, which Nouwen 2010 calls “Class A” and “Class B” modifiers respectively. Class A modifiers include, in addition to the comparatives in (2), the prepositional modifiers *over*, *under* and *between* __ and __; Class B modifiers include, in addition to the superlatives in (3), the adverbs *minimally*, *maximally*, the prepositions *from*, *up to*, and complex expressions like __ or *more/fewer*. The two classes are internally heterogeneous, with particular modifiers having slightly different distributional and semantic properties (see e.g. Nouwen 2008b, Schwarz, Buccola & Hamilton 2012, Rett 2014), but there are two core semantic features that the members of each class respectively share, and which distinguish one class from the other:

- (4) a. Class A modifiers express exclusive (strict) orderings relative to the modified numeral.
- b. Class B modifiers express inclusive (non-strict) orderings relative to the modified numeral.

The goal of this paper is to argue that it is this distinction in core meaning that determines the presence vs. absence of ignorance inferences. In line with the other researchers listed above, I will argue that ignorance inferences are pragmatic in nature, and arise because Class B modified numerals are less informative than relevant alternatives. Specifically, I will argue that the alternatives for a Class B modified numeral are the corresponding Class A modified numeral and the bare numeral, with the latter analyzed as a generalized quantifier over degrees that introduces “two-sided” truth-conditional meaning as a matter of semantic content (Kennedy 2013). On this view, (3a-b) are asymmetrically entailed by (2a-b) (respectively) and (1), so an utterance of the former implicates that the speaker does not know whether the latter holds under standard Quantity reasoning, in exactly the same way that, for

instance, the utterance of a disjunction implicates that the speaker does not know of either disjunct whether it holds.

The paper is organized as follows. I begin with a discussion of the analysis of Class A/B modifiers in [Nouwen 2010](#), which is also based on the core semantic distinction in (4) but derives ignorance implications semantically rather than pragmatically. Even though I will ultimately conclude that Nouwen’s analysis is empirically problematic, it is important because it highlights several interactions between modified numerals and modals that any analysis needs to explain. I will then turn to my own proposal, which begins from the semantic analysis of unmodified numerals defended in [Kennedy 2013](#), which analyzes them as generalized quantifiers over degrees that introduce two-sided truth conditional content. I argue that this semantic analysis of bare numerals provides the basis for a pragmatic theory in which the alternatives that are relevant for the calculation of Quantity implicatures of utterances of sentences containing numerals are just the ones we need in order to generate ignorance inferences with Class B modifiers but not with Class A modifiers. I then show how the analysis accounts for the interaction of modified numerals and modals, with particular attention to cases in which ignorance implications disappear ([Büring’s \(2008\)](#) “authoritative readings”).

2 Nouwen’s analysis of the Class A/B distinction

2.1 Blocking and rescuing

Nouwen’s account of how the semantic difference between Class A and Class B modifiers relates to ignorance implications has three parts. The first is the hypothesis that numerals (both modified and unmodified) saturate a degree position in the nominal projection which, following [Hackl 2000](#), he takes to be provided by a parameterized cardinality determiner *many*. This determiner comes in two versions: the “weak” version in (5a), which involves regular existential quantification over plural individuals; and the “strong” version in (5b) which adds a uniqueness requirement (indicated by “!”).

- (5) a. $\llbracket \text{many}_w \rrbracket = \lambda n \lambda P \lambda Q. \exists x [Q(x) \wedge P(x) \wedge \#(x) = n]$
 b. $\llbracket \text{many}_s \rrbracket = \lambda n \lambda P \lambda Q. \exists ! x [Q(x) \wedge P(x) \wedge \#(x) = n]$

The second part involves a semantic distinction between modified and unmodified numerals. The latter are singular terms, and denote values in the range of the measure function “#” encoded by *many*, which Nouwen (with

Hackl) assumes to be elements in the domain of degrees (see also Cresswell 1976, Krifka 1989).¹ Depending on which version of *many* is chosen, the result is either lower-bounded (*many_w*) or two-sided (*many_s*) truth conditions, as illustrated in (6), the two parses of (1).

- (6) a. $\exists x[\text{have}(x)(\text{airplane}) \wedge \text{exits}(x) \wedge \#(x) = 6]$
 b. $\exists!x[\text{have}(x)(\text{airplane}) \wedge \text{exits}(x) \wedge \#(x) = 6]$

Modified numerals, on the other hand, denote generalized quantifiers over degrees. Nouwen assumes a fairly standard semantics for Class A modifiers, which builds on work in comparatives: *more than* and *fewer than* have the denotations in (7a–b).

- (7) a. $\llbracket \text{more than} \rrbracket = \lambda m \lambda P_{\langle d,t \rangle}. \max\{n \mid P(n)\} > m$
 b. $\llbracket \text{fewer than} \rrbracket = \lambda m \lambda P_{\langle d,t \rangle}. \max\{n \mid P(n)\} < m$

These denotations give the intuitively correct truth conditions for (2a–b) in (8a–b): the maximum number of sides that the airplane has is greater than six and fewer than six, respectively. Here there is no truth conditional difference here between *many_w* and *many_s*, so I just give meanings for the former.

- (8) a. $\max\{n \mid \exists x[\text{have}(x)(\text{airplane}) \wedge \text{exits}(x) \wedge \#(x) = n]\} > 6$
 b. $\max\{n \mid \exists x[\text{have}(x)(\text{airplane}) \wedge \text{exits}(x) \wedge \#(x) = n]\} < 6$

For the Class B modifiers, Nouwen proposes the meaning in (9a) for lower-bound modifiers like *at least*, and the one in (9b) for upper-bound modifiers like *at most*.

- (9) a. $\llbracket \text{at least} \rrbracket = \lambda m \lambda P_{\langle d,t \rangle}. \min\{n \mid P(n)\} = m$
 b. $\llbracket \text{at most} \rrbracket = \lambda m \lambda P_{\langle d,t \rangle}. \max\{n \mid P(n)\} = m$

These denotations give the truth conditions in (10a–b) for (3a–b), respectively. For the lower-bound Class B modifiers, it is now crucial that we parse the sentence using *many_s*, because *many_w* returns contradictory truth conditions for any numeral greater than one.

¹ Note that # is not, strictly speaking, a cardinality function, but rather gives a measure of the size of the (plural) individual argument of the noun in “natural units” based on the sense of the noun (Krifka 1989, Salmon 1997). If this object is formed entirely of atoms, then # returns a value that is equivalent to a cardinality. But if this object contains parts of atoms, then # returns an appropriate fractional or decimal measure. For the purpose of this paper, we can assume that the range of # has at least the structure of the real numbers (Fox & Hackl 2007).

- (10) a. $\min\{n \mid \exists x![\text{have}(x)(\text{airplane}) \wedge \text{exits}(x) \wedge \#(x) = n]\} = 6$
 b. $\max\{n \mid \exists x[\text{have}(x)(\text{airplane}) \wedge \text{exits}(x) \wedge \#(x) = n]\} = 6$

Both (10a) and (10b) introduce two-sided truth conditions: the former requires the minimum unique plurality of airplane exits to be exactly six; the latter requires the maximum unique plurality of airplane exits to be exactly six. Observing that these truth conditions are equivalent to what we get from corresponding, simpler, bare numeral constructions on their *many_s* parses, Nouwen proposes that principles of blocking rule them out as possible meanings for (3a-b).

However, there is a way of modifying (10a-b) to derive interpretations for (3a-b) that are distinct from the bare numeral meanings, which is the third part of Nouwen's account of the ignorance implications associated with Class B modifiers. Nouwen proposes that a silent epistemic possibility modal can be inserted into sentences like those in (3). If the modified numeral takes scope over the modal, we get the truth conditions in (11a-b), which differ crucially from those in (10a-b) in that the sets that are the inputs to the *min* and *max* operators are no longer singletons.

- (11) a. $\min\{n \mid \diamond \exists x![\text{have}(x)(\text{airplane}) \wedge \text{exits}(x) \wedge \#(x) = n]\} = 6$
 b. $\max\{n \mid \diamond \exists x[\text{have}(x)(\text{airplane}) \wedge \text{exits}(x) \wedge \#(x) = n]\} = 6$

The formula in (11a) says that the minimum number in the set of unique numbers n such that there is an epistemically accessible world in which the airplane has n emergency exits is six; that in (11b) says that the maximum such number is six. When there is uncertainty about quantity, these sets will contain more than one number, since uncertainty entails the existence of epistemically accessible worlds in which the airplane has a different number of emergency exits. The contribution of the modifiers in such a situation is to express lower and upper bounds, respectively, on the relevant sets, and the result is exactly the uncertainty implications that we wanted to derive.

2.2 Two problems with modals

Although Nouwen's analysis succeeds in deriving the uncertainty implications of Class B modifiers, it faces a number of empirical challenges. Some of these have already been discussed in the literature (see [Schwarz, Buccola & Hamilton 2012](#)); here I want to focus on two problems involving the interaction of Class B modified numerals and modals that raise further challenges which, I believe, point to the need for a new account of the Class A/B distinction.

2.2.1 Class B modifiers and deontic modals

The first problem with modals is noted by Nouwen himself: the analysis appears to make incorrect predictions about the truth conditions of sentences in which Class B modified numerals are embedded under deontic modals. Nouwen focuses on the case of (12), which is predicted to have the two interpretations shown in (12a–b), depending on whether the numeral takes scope below or above the modal. (In the examples to follow, I use “ \otimes ” to indicate a reading whose unavailability can be explained in Nouwen’s account by blocking, and “ $*$ ” to indicate a reading whose unavailability cannot be so explained.)

(12) You are required to register for at least three classes.

- a. $\otimes \Box[\min\{n \mid \exists!x[\text{reg}(x)(\text{you}) \wedge \text{classes}(x) \wedge \#(x) = n]\} = 3]$
- b. $* \min\{n \mid \Box\exists!x[\text{reg}(x)(\text{you}) \wedge \text{classes}(x) \wedge \#(x) = n]\} = 3$

On the narrow scope numeral interpretation, (12) is predicted to be true just in case every deontically accessible world is one in which the minimum unique number of classes registered for is three, which entails registration in no more and no fewer than three classes. The sentence in (12) clearly does not have such an interpretation, though the corresponding bare numeral sentence does (*You are required to register for three courses*), so the absence of this reading could be explained in terms of blocking. The problem is that the truth conditions associated with the wide scope numeral interpretation are identical: (12b) says that three is the minimum n such that in every deontically accessible world there is registration in a unique number of classes equal to n , which again entails registration in no more and no fewer than three classes. So this reading should be blocked as well, and the sentence should be unacceptable.

Of course, (12) is perfectly acceptable, and its truth conditions are clear: registration in one or two (or zero) classes is not allowed; registration in three is a must; registration in more than three is an option. Moreover, there need be no speaker ignorance: (12) can be understood as an “authoritative” statement about course enrollment requirements, to use the terminology of [Büring 2008](#). As Nouwen observes, this meaning is represented by the formula in (13), in which the numeral scopes over an existential modal operator.

(13) $\min\{n \mid \Diamond\exists!x[\text{reg}(x)(\text{you}) \wedge \text{classes}(x) \wedge \#(x) = n]\} = 3$

Nouwen does not provide an explanation for how or why this shift from universal to existential modal force occurs, or why there is not another reading in which the modified numeral scopes below the modal (which would give the wrong truth conditions), though he notes that it appears to be a systematic feature of universal modal statements expressing minimum requirements (cf. von Fintel & Iatridou 2007), showing up also in relative clause structures like (14).

(14) The minimum number of classes that you need to register for is three.

The analysis that I will present in Section 3.2 will derive the correct meanings for these sentences without invoking blocking effects or necessitating a change in modal force.

Nouwen focuses mainly on the interaction of minimizing Class B modifiers with universal modals, but when we look at the full range of possibilities — pitting modal force against maximality/minimality — we see that the problems go beyond the examples just considered. Consider first the interaction of maximizing modifiers and universal modals. The following example is predicted to have the two readings in (15a-b) depending on whether the numeral scopes below or above the modal, neither of which are correct.

(15) You are required to register for at most three classes.

- a. $\otimes \Box[\max\{n \mid \exists x[\text{reg}(x)(\text{you}) \wedge \text{classes}(x) \wedge \#(x) = n]\} = 3]$
- b. $* \max\{n \mid \Box[\exists x[\text{reg}(x)(\text{you}) \wedge \text{classes}(x) \wedge \#(x) = n]]\} = 3$

The formula in (15a) says that every deontically accessible world is such that there is a maximum of three courses registered for. If three is the maximum in every world, then there are no worlds with registration in one or two classes, and no worlds with registration in four or more classes; that is, there is registration in exactly three courses in all worlds. This is not a reading of (15), but Nouwen could again appeal to blocking here; (15b), however, presents a bigger problem. This logical representation says that three is the maximum n such that in every world, there is registration for at least n classes. This disallows registration in one or two classes, and allows registration in three or more classes; in other words, this is precisely the meaning that we failed to derive without modal shifting for (15)! It is, moreover, one of the meanings associated with the corresponding bare numeral sentence, namely the lower-bounded one (Geurts 2006), but it is not a possible meaning of (15).

In fact, (15) is ambiguous, but neither reading is captured by (15a-b). One reading forbids enrollment in four or more classes, and does not introduce a

speaker ignorance implication; this is Büring’s (2008) authoritative reading. The second reading is weaker, and includes an ignorance implication: registration in one to three classes is required, but the speaker does not know which, and registration in four or more classes is not ruled out; this is what Büring 2008 calls the “speaker insecurity” reading. This latter reading can be derived in Nouwen’s analysis by inserting an existential epistemic modal between the numeral and the deontic modal, but it is not clear how to capture the authoritative reading.

Next consider minimizing Class B modifiers and existential root modals. The following sentence should have the readings represented in (16a-b).

- (16) You are allowed to register for at least three classes.
- a. $\otimes \diamond[\min\{n \mid \exists!x[\text{reg}(x)(\text{you}) \wedge \text{classes}(x) \wedge \#(x) = n]\} = 3]$
 - b. $* \min\{n \mid \diamond[\exists!x[\text{reg}(x)(\text{you}) \wedge \text{classes}(x) \wedge \#(x) = n]]\} = 3$

The formula in (16a) says that there is a deontically accessible world with registration in exactly three classes. This is a possible understanding of (16), but it is also one of the meanings assigned to the corresponding bare numeral sentence, and so (on Nouwen’s account) it should be blocked. The one in (16b) says that three is the minimum n such that there is a deontically accessible world with exactly n registered-for classes. This allows for worlds with registration in more than three classes, but it rules out worlds with registration in one or two classes. The expectation, then, is that (16) should have the meaning that we wanted to derive for (12). This is actually not surprising, given Nouwen’s suggestion that the universal modal force is converted to existential modal force in that example, but it is a problem for the analysis of (16), since it cannot be understood in this way.

Finally, consider the interaction of existential modals and maximizing Class B modifiers.

- (17) You are allowed to register for at most three classes.
- a. $\otimes \diamond[\max\{n \mid \exists x[\text{reg}(x)(\text{you}) \wedge \text{classes}(x) \wedge \#(x) = n]\} = 3]$
 - b. $\max\{n \mid \diamond[\exists x[\text{reg}(x)(\text{you}) \wedge \text{classes}(x) \wedge \#(x) = n]]\} = 3$

If the numeral scopes below the modal, as in (17a), the predicted meaning is that registration in exactly three courses is allowed. This is a weak meaning, because it does not rule out registration in four or more courses (or in one or two courses). It does not appear to be a possible reading of (17), which explicitly puts an upper bound on the number of registered-for courses,

though it is a meaning of the corresponding bare numeral sentence, so Nouwen can appeal to blocking here.

If the numeral scopes above the modal, as in (17b), the truth conditions state that three is the maximal n such that there is a deontically accessible world with registration in n classes. This is precisely what (17) means, so it appears that Nouwen's analysis derives the correct result for this case. However, it is also the case that the bare numeral sentence (18) can be used to convey precisely the same information, that is, that registration in more than three courses is not allowed:

(18) You are allowed to register for three classes.

Why, then, is there no blocking effect here? Nouwen points out that, given his assumptions about bare numerals, this meaning is only derivable by implicature, and assumes that only truth-conditional content is relevant for triggering blocking effects. The formulas in (19a-b) show the truth conditions that Nouwen predicts for (18), which differ in whether the propositional argument of the modal has a one-sided (*many_w*) or two-sided (*many_s*) understanding of the numeral construction.

- (19) a. $\diamond[\exists x[\text{reg}(x)(\text{you}) \wedge \text{classes}(x) \wedge \#(x) = n]]$
 b. $\diamond[\exists!x[\text{reg}(x)(\text{you}) \wedge \text{classes}(x) \wedge \#(x) = n]]$

The formula in (19b) is equivalent to (17a), simply entailing the existence of a world with registration in exactly three classes, but not ruling out worlds with registration in other numbers of classes. The one in (19a) is even weaker, since we only know that registration in some number of classes of at least size three is acceptable. But precisely because both readings are so weak, they can be pragmatically strengthened by a Quantity implicature of the sort in (20).

(20) $\forall n > 3 \neg[\diamond[\exists(!)x[\text{reg}(x)(\text{you}) \wedge \text{classes}(x) \wedge \#(x) = n]]]$

Adding (20) to the semantic content of (18) (on either interpretation) derives a meaning that is equivalent to (17b), but since (20) is an implicature, this (by hypothesis) does not block (17b).

That said, if we had reason to believe that the content in (20) were part of the truth conditions of (18), then presumably we *would* expect blocking here, and Nouwen's account of the meaning and acceptability of (17) would no longer be tenable. In Section 3.1, I will present such evidence, and provide

A “de-Fregean” semantics for modified and unmodified numerals

a semantic analysis of unmodified numerals that derives upper-bounded readings compositionally. If this analysis is correct, then Nouwen’s account of (17) cannot be maintained.

2.2.2 Class B modifiers and epistemic modals

I now turn to interactions with overt epistemic modals, which are not discussed by Nouwen and which contrast in a significant way with the covert epistemic modals that are so crucial to the derivation of ignorance implications. First consider the following two examples, which contain minimizing and maximizing Class B modifiers, respectively.

- (21) a. Chicago has at least 200 distinct neighborhoods.
b. Chicago has at most 3,000,000 residents.

As expected, these sentences are acceptable only if there is some uncertainty about the exact number of neighborhoods in Chicago, and about the exact population in Chicago; when uncertainty is eliminated, the sentences are infelicitous:

- (22) #Thanks to the detailed information provided in this census report, I know precisely how many distinct neighborhoods Chicago has, and its exact population: it has at least 200 distinct neighborhoods, and it has at most 3,000,000 residents.

Recall from the discussion in Section 2 that Nouwen derives the uncertainty inferences in (21) by scoping the modified numeral over an unpronounced epistemic possibility modal, which is itself inserted into the structure to bypass the blocking effect that would otherwise arise from the truth-conditional equivalence between the non-modalized versions of (21a-b) and the corresponding sentences with bare numerals. This account of the uncertainty inference would seem to predict that variants of (21) with *overt* epistemic modals, such as (23a-b), should have parallel meanings.

- (23) a. Chicago might have at least 200 distinct neighborhoods.
b. Chicago might have at most 3,000,000 residents.

But this is not the case: (23a-b) are not synonymous with (21a-b), and in particular, they lack interpretations in which the modified numeral takes scope over the modal. (23a-b) can only be understood to say that it is epistemically

possible that the minimum number of Chicago neighborhoods is 200 and that it is epistemically possible that its maximum population is 3,000,000; these sentences cannot be understood to make the stronger claims associated with (21a–b) which rule out epistemically accessible worlds in which Chicago has fewer than 200 distinct neighborhoods or more than 3,000,000 residents.

In fact, the absence of a wide-scope interpretation for the modified numeral in these examples is expected: previous work on degree quantifiers — expressions of type $\langle\langle d, t \rangle, t\rangle$, which includes the class of modified numerals in Nouwen’s analysis — has shown that they may take scope higher than root modals but not higher than epistemic modals (Heim 2000, Schwarzschild & Wilkinson 2002, Buring 2007, Krasikova 2010, Alrenga & Kennedy 2014). This is illustrated by the following pair of examples, which involve the negative differential degree quantifier *no*:

- (24) a. Kim can jump no higher than Lee.
 b. Kim might jump no higher than Lee.

The sentence in (24a) has a reading in which it asserts that Kim lacks the ability to jump higher than Lee (and implicates that Kim can jump at least as high as Lee). Alrenga & Kennedy 2014 show that this reading arises when *no* takes scope over the ability modal, deriving the truth conditions stated informally in (25).

$$(25) \quad \{d \mid \Diamond[\text{Kim jumps } d\text{-higher than Lee in } w]\} = \emptyset$$

When the relevant accessibility relation is based on Kim’s abilities, this derives the observed interpretation of (24a). What is important for us here is that (24b) does not have a corresponding wide-scope meaning for *no*. Such a meaning would say that the set of degrees *d* such that there is an epistemically accessible world in which Kim jumps *d*-higher than Lee is empty, which is another way of saying that there is no evidence that Kim jumped higher than Lee. But (24b) merely says that there is evidence that Kim’s jump will not be higher than Lee’s, which is a much weaker claim, and is precisely the one that we get when *no* stays within the scope of the modal:

$$(26) \quad \Diamond[\{d \mid \text{Kim jumps } d\text{-higher than Lee in } w\} = \emptyset]$$

If it is generally the case that degree quantifiers cannot take scope over epistemic modals, then the missing readings of (23) are expected. On the other hand, if it is generally the case that degree quantifiers cannot take scope

over epistemic modals, then Nouwen’s account of uncertainty inferences with Class B modifiers faces a serious challenge, since it rests on the idea that precisely such a scopal configuration is created in order to avoid a blocking effect and generate the observed interpretations of sentences like the ones in (21a–b). Indeed, to derive the actual meanings of (23a–b), Nouwen must assume that in addition to the overt modal, a second, silent modal is inserted below the scope of the modified numeral, as shown in (27a–b).

- (27) a. $\diamond[\min\{n \mid \diamond[\exists!x[\text{have}(x)(\text{chi}) \wedge \text{neighborhoods}(x) \wedge \#(x) = n]]\} = 200]$
 b. $\diamond[\max\{n \mid \diamond[\exists!x[\text{have}(x)(\text{chi}) \wedge \text{residents}(x) \wedge \#(x) = n]]\} = 3,000,000]$

In the absence of the silent modal, the truth conditions for (23a–b) would be identical to corresponding sentences with bare numerals, for the reasons described above, and so should be blocked.

There are various ways that Nouwen could respond to this challenge. Perhaps “last resort” insertion of a silent epistemic modal does not interact with whatever constraint blocks wide scope of a degree quantifier relative to an existential modal. Or perhaps the observed readings are not compositionally derived in the first place, but rather involve some sort of pragmatic mapping from the compositionally-derived content (which is ruled out by blocking) to the observed meaning. On the other hand, if an alternative analysis avoids these complications entirely, that would certainly be a point in its favor. I now turn to such an analysis.

3 A “de-Fregean” semantics for modified and unmodified numerals

In this section, I present an alternative account of the properties of Class A/B modified numerals, which is designed to account for both ignorance implications with Class B modifiers and for the interactions with modals that we examined in Section 2.2. The central thesis is that all numerals — modified and unmodified alike — have essentially the same core meanings: they denote second order properties of degrees, differing only in the kinds of orderings they introduce. In Section 3.1, I present my account of unmodified numerals, and then extend the analysis to modified numerals in Section 3.2. I argue that the uncertainty implications of Class B modifiers arise as implicatures because they are less informative than either Class A modified numerals or unmodified numerals, and I show that this analysis avoids the problems with

modals that Nouwen’s account runs into, deriving the correct meanings for the various scopal possibilities.

3.1 Unmodified numerals

For many years, the standard view about the relation between sentence meaning and speaker meaning in assertions involving numerals was the one expressed in the following quote from [Horn 1972](#):

Numbers, then, or rather sentences containing them, assert lower boundedness — *at least n* — and given tokens of utterances containing cardinal numbers may, depending on the context, implicate upper boundedness — *at most n* — so that the number may be interpreted as denoting an exact quantity.

([Horn 1972](#), p. 33)

Although this view is still encountered in the literature and in introductory semantics and pragmatics courses, a range of studies have appeared since [Horn 1972](#) in both the theoretical and experimental literature (some by Horn himself) which provide compelling evidence that the two-sided “exactly” understanding of sentences containing numerals is a matter of semantic content, rather than implicature (see e.g. [Sadock 1984](#), [Koenig 1991](#), [Horn 1992](#), [Scharten 1997](#), [Carston 1998](#), [Krifka 1998](#), [Noveck 2001](#), [Papafragou & Musolino 2003](#), [Bultinck 2005](#), [Geurts 2006](#), [Breheny 2008](#), [Kennedy 2013](#)).

The most straightforward way to implement a two-sided semantics for sentences with numerals is to bite the bullet and analyze numerals as quantificational determiners with two-sided meanings, as in (28a) ([Koenig 1991](#), [Breheny 2008](#)), or as determiners that are ambiguous between the two-sided meaning in (28a) and the lower-bounded meaning in (28b) ([Geurts 2006](#)).

- (28) a. $\llbracket \text{three} \rrbracket = \lambda P_{\langle e,t \rangle} \lambda Q_{\langle e,t \rangle} . |P \cap Q| = 3$
 b. $\llbracket \text{three} \rrbracket = \lambda P_{\langle e,t \rangle} \lambda Q_{\langle e,t \rangle} . |P \cap Q| \geq 3$

A second way is to adopt the assumptions that I outlined in the previous section: treat bare numerals as singular terms denoting numbers (objects of type d), and assume two versions of Hackl’s (2000) parameterized cardinality determiner, $many_s$ and $many_w$, which differ in whether they introduce simple existential quantification over individuals or existential quantification

plus uniqueness. Composition of a numeral with the former derives lower-bounded semantic content; composition with the latter derives two-sided semantic content.

In Kennedy 2013, however, I develop an alternative analysis of two-sided content which is based on the hypothesis that unmodified numerals, just like modified numerals in Nouwen’s (2010) analysis, can denote second order properties of degrees. Specifically, I propose that unmodified numerals have type $\langle\langle d, t \rangle, t\rangle$ generalized quantifier denotations of the sort shown in (29).

$$(29) \quad \llbracket \text{three} \rrbracket = \lambda D_{\langle d, t \rangle}. \max\{n \mid D(n)\} = 3$$

The numeral *three*, on this analysis, is true of a property of degrees if the maximum number that satisfies the property is three. This analysis is inspired by the treatment of numerals as second order properties of individuals considered in Frege 1980 [1884], in which for instance *three* is true of a property of individuals just in case the number of individuals that the property is true of is three (cf. Scharten 1997), and so I refer to it as a “de-Fregean” semantics for numerals. The central compositional difference between the two accounts is that I treat bare numerals as members of the class of degree expressions and numbers as degrees, and so am able to make full use of all the principles and assumptions of degree syntax and semantics that have been established in work on comparatives, modified numerals and other degree constructions.

In particular, as in Nouwen’s analysis, I assume that numerals saturate a degree position in the nominal projection, which could be introduced by Hackl’s *many_w* (we no longer need *many_s*, as we will shortly see), by a silent or deleted adjectival version of *many* (cf. Landman 2003, 2004), or by the noun itself (Cresswell 1976, Krifka 1989). For present purposes, these distinctions do not matter, so I will adopt the Cresswell/Krifka approach for simplicity, and assume that the nominal and verb combine via Chung & Ladusaw’s (2004) Restrict rule, with the individual argument of the noun bound by existential closure. The result is that a sentence like (30a) has the truth conditions in (30b) when *three* is interpreted as a de-Fregean quantifier.

- (30) a. Kim took three classes.
 b. $\max\{n \mid \exists x[\text{took}(x)(\text{kim}) \wedge \text{classes}(x) \wedge \#(x) = n]\} = 3$

According to (30b), (30a) is true just in case the maximal n such that Kim took at least n classes is equal to three, which is false if she took two and false if she took four. These are the two-sided truth conditions.

One of the main advantages of the de-Fregean analysis, in addition to deriving two-sided meanings semantically, is that it explains a pattern of scalar readings first (to my knowledge) observed by [Scharten 1997](#), and illustrated very clearly in experimental work by [Musolino 2004](#). Specifically, although two-sided readings are the default for simple sentences like (30a), one-sided (lower- and upper-bounded) readings appear in a systematic and predictable way when numerals are embedded under (root) modals: lower-bounded readings appear when a numeral is embedded under a universal modal, and upper-bounded readings emerge when a numeral is embedded under an existential modal. The examples in (31) and (32) illustrate the pattern.

- (31) a. In Britain, you have to be 17 to drive a motorbike and 18 to drive a car.
 b. Mary needs three As to get into Oxford.
 c. Goofy said that the Troll needs to put two hoops on the pole in order to win the coin.
 d. You must provide three letters of recommendation.
 e. You are required to take three classes per quarter.
- (32) a. She can have 2000 calories a day without putting on weight.
 b. You may have half the cake.
 c. Pink panther said the horse could knock down two obstacles and still win the blue ribbon.
 d. You are permitted to take three cards.
 e. You are allowed to enroll in three classes per quarter.

The de-Fregean analysis derives this pattern as a scopal interaction between numerals and modals. (For independent arguments that bare numerals must be able to take scope separately from the nominals with which they compose in the surface form, see [Kennedy & Stanley 2009](#).)

Consider first the case of universal modals. The sentence in (33) can be interpreted either with the number word inside the scope of the modal, deriving the proposition in (33a), or with the modal inside the scope of the number word, deriving the proposition in (33b).

- (33) Kim is required to take three classes.
- a. $\Box[\max\{n \mid \exists x[\text{take}(x)(\text{kim}) \wedge \text{classes}(x) \wedge \#(x) = n]\} = 3]$
 b. $\max\{n \mid \Box[\exists x[\text{take}(x)(\text{kim}) \wedge \text{classes}(x) \wedge \#(x) = n]]\} = 3$

The proposition in (33a) is true just in case every deontically accessible world is such that the maximum number of classes taken by Kim in that world is three. This is the two-sided reading. The proposition in (33b) is true just in case the maximum number n , such that in every deontically accessible world there is a plurality of classes of at least size n taken by Kim, is three. This entails that the minimum number of deontically acceptable classes is three, which is the lower bounded meaning.

In the case of a sentence with an existential modal like (34), we get exactly the same scopal relations, but the resulting truth conditions are quite different:

- (34) Kim is allowed to take three classes.
- a. $\diamond[\max\{n \mid \exists x[\text{take}(x)(\text{kim}) \wedge \text{classes}(x) \wedge \#(x) = n]\} = 3]$
 - b. $\max\{n \mid \diamond[\exists x[\text{take}(x)(\text{kim}) \wedge \text{classes}(x) \wedge \#(x) = n]]\} = 3$

The proposition in (34a) is the “weak” reading of (34), which merely says that there is a deontically accessible world in which the maximum number of classes taken by Kim is three. The one in (34b), on the other hand, says that the maximum n such that there is a deontically acceptable world in which Kim takes at least n classes is three. On this reading, the sentence is false if there is a deontically accessible world in which Kim takes more than three classes. This is the “strong” reading of (34), and the fact that it is derived compositionally, rather than via a scalar implicature, represents one of the central empirical differences between the de-Fregean analysis and all other approaches to number word meaning, in which such readings can only be derived via implicature.

We have already seen that an analysis of two-sided readings in terms of *many_s* cannot derive upper-bounded readings in examples like (34) semantically, because embedding a *many_s* proposition under an existential modal, as in (35a), only derives the weak reading of (34). Similarly, embedding a Breheny-style two-sided quantificational determiner under the modal, as in (35b), derives only the weak reading.

- (35) a. $\diamond[\exists!x[\text{take}(x)(\text{kim}) \wedge \text{classes}(x) \wedge \#(x) = n]]$
 b. $\diamond[|\lambda x.\text{classes}(x) \cap \lambda x.\text{take}(x)(\text{kim})| = 3]$

Of course, both of these meanings can be straightforwardly strengthened to produce the strong reading by adding a scalar implicature to the effect that there is no world w such that the number of classes in w is exactly n ,

for all n greater than three; so the crucial question is whether the upper-bounded reading can be retained in environments in which scalar implicatures disappear. If the answer is yes, then we know that it must be derived as a matter of semantic content, and we have evidence in support of the de-Fregean analysis over the existing alternatives.

Such evidence is discussed in [Kennedy 2013](#). To set up the example, consider a scenario involving three different groups of taxpayers who can be distinguished according to how many exemptions they are allowed to claim on their tax returns, where the minimum number allowed by law is zero and the maximum number allowed by law is four. Group A contains individuals who are not allowed to claim any exceptions at all; Group B contains individuals who are not allowed to claim more than two exemptions; and Group C contains individuals who are allowed to claim all four exemptions. The society in which these individuals live is exemption-maximizing but law-abiding, so everyone in Group A claims zero exemptions, everyone in Group B claims exactly two exemptions, and everyone in group C claims exactly four exemptions. Now consider the following utterances as descriptions of this situation:

- (36) a. No individual who was allowed to claim two exemptions claimed four.
 b. No individual who was allowed to claim some exemptions claimed four.

The sentence in (36a) has a reading in which it is true in this scenario, because the quantifier restriction is understood to pick out individuals who were allowed to claim two exemptions, and not allowed to claim more than two exemptions; that is, the ones in Group B. This is an upper-bounded reading, but it occurs in a downward-entailing context (and in the argument of a logical operator), which is a context in which scalar implicatures are suppressed. And indeed, (36b), in which the number word is replaced by the scalar quantifier *some*, is false in this situation, because the quantifier is understood to range over all individuals who were allowed to claim exemptions, which includes the ones in Group C. Unless *only* is inserted or *some* is pronounced with strong phonological prominence, neither of which is necessary in (36a), this sentence does not have a reading in which the restriction ranges

over individuals who were not allowed to claim all exemptions, which would exclude the individuals in Group C and make it true.²

Before turning to the analysis of modified numerals, let me say a few words about the relation between the de-Fregean quantificational meaning of a number word and the singular term meaning. For the purposes of this paper, it would be fine to assume that an unmodified numeral like *three* has only the de-Fregean denotation in (29), since this is the meaning that will play a role in accounting for the patterns of interpretations that we find with Class A and Class B modified numerals. However, even though there is good reason to believe that the two-sided truth conditions that this meaning introduces are both a matter of semantic content and also a strong default, examples like the following show that there is still reason to think that lower-bounded, one-sided meanings are available as well:

- (37) a. Kim took three classes, if not four.
 b. No one who misses three questions on the exam will receive a drivers license.

If the first part of (37a) had only the two-sided interpretation in (30b), then the continuation in the second part should sound strange (Horn 1972). Similarly, the domain of the quantifier in (37b) is most naturally understood to be those people who miss at least three questions on the exam. These examples appear to directly contradict the hypothesis that numerals introduce two-sided truth-conditional content.

In fact, there is a way to derive one-sided content for these examples, even if the de-Fregean meaning of a numeral is basic, as I have proposed. A singular term meaning for the numeral can be derived from the de-Fregean meaning by successive application of Partee’s (1987) BE and iota operations, defined in (38).

- (38) a. $BE = \lambda Q_{\langle\langle\alpha,t\rangle,t\rangle} \lambda x_{\alpha} . Q(\lambda y_{\alpha} . y = x)$
 b. $iota = \lambda P_{\langle\alpha,t\rangle} . \iota x_{\alpha} [P(x)]$

² The contrast in (36) also argues in favor of the de-Fregean analysis over an exhaustivity-based account of two-sided readings of bare numerals, since the latter view would predict parallel behavior of numerals and other scalar terms. In fact, numerals systematically differ from other scalar terms in retaining two-sided (“exhaustified”) meanings in contexts that tend to suppress scalar implicatures, such as the downward-entailing environment in (36) (see e.g. Koenig 1991, Horn 1992, Marty, Chemla & Spector 2013, Kennedy 2013).

BE maps a generalized quantifier to the property that is true of all the singletons in the denotation of the quantifier. Application of BE to the de-Fregean denotation of *three* derives the property of being a number equal to three, as shown by the following derivation:

$$\begin{aligned}
 (39) \quad & \text{BE}(\text{three}) \\
 & = [\lambda Q \lambda m. Q(\lambda p. p = m)](\lambda P. \max\{n \mid P(n)\} = 3) \\
 & = [\lambda m. [\lambda P. \max\{n \mid P(n)\} = 3](\lambda p. p = m)] \quad (\lambda\text{-conversion}) \\
 & = \lambda m. \max\{n \mid [\lambda p. p = m](n)\} = 3 \quad (\lambda\text{-conversion}) \\
 & = \lambda m. \max\{n \mid n = m\} = 3 \quad (\lambda\text{-conversion}) \\
 & = \lambda m. m = 3 \quad (\text{substitution of equivalents})
 \end{aligned}$$

Application of *iota* to the result returns the unique number equal to three, which is of course three itself.³ And given the possibility of lowering *three* to a singular term meaning, we now predict a systematic ambiguity for numerals. In the case of (37a), for example, the de-Fregean meaning derives the two-sided interpretation in (40a), and the type-lowered singular term meaning gives the one-sided, lower-bounded truth conditions in (40b).

- (40) Kim took three classes.
- a. $\max\{n \mid \exists x[\text{took}(x)(\text{kim}) \wedge \text{classes}(x) \wedge \#(x) = n]\} = 3$
 - b. $\exists x[\text{took}(x)(\text{kim}) \wedge \text{classes}(x) \wedge \#(x) = 3]$

It might seem counterintuitive to assume that it is the de-Fregean, quantificational meaning of the numeral that is basic, rather than the simpler singular term denotation, but there are a few reasons to think that this is correct. First, from a purely theoretical perspective, the latter can be derived from the former using Partee's type-shifting principles, but the former cannot be so derived from the latter. Instead, it can only be derived through the addition of some meaning changing operation, such as exhaustification.

³ In fact, if the grammar includes an operation like Chung & Ladusaw's Restrict, as I have assumed, then we don't actually need to map the de-Fregean meaning to a singular term in order to derive the one-sided interpretation of (37a); the $\langle d, t \rangle$ meaning in (39) is enough. Composition of this meaning with the type $\langle d, \langle e, t \rangle \rangle$ noun (or implicit cardinality predicate) meaning plus existential closure over the degree argument derives (i), which is equivalent to (40b).

(i) $\exists x \exists n[\text{took}(x)(\text{kim}) \wedge \text{classes}(x) \wedge \#(x) = n \wedge n = 3]$

Second, the evidence that has been presented both in traditional linguistic investigations and in experimental and developmental work over the past twenty or so years overwhelmingly favors the view that the default interpretation of simple sentences with bare numerals is the two-sided one, with the one-sided meaning arising only in special contexts (Koenig 1991, Horn 1992, Scharten 1997, Musolino 2004, Bultinck 2005, Geurts 2006, Hurewitz et al. 2007, Huang, Spelke & Snedeker 2013, Marty, Chemla & Spector 2013, Kennedy 2013). Finally, as we will see in the next section, the hypothesis that the basic meaning of an unmodified numeral is the two-sided, de-Fregean one plays a crucial role in the pragmatic analysis of ignorance implications with modified numerals.

3.2 Modified numerals

With this account of unmodified numerals in hand, we can now turn to the analysis of modified numerals, and the explanation for the patterns of data discussed in Section 2. For Class A modifiers like *more than* and *fewer than*, I will assume with Nouwen 2010 that they combine with the singular term denotation of a numeral to give back a generalized quantifier over degrees with a comparative meaning, as in (41).

- (41) a. $\llbracket \text{more than} \rrbracket = \lambda n \lambda P_{\langle d,t \rangle} . \max\{n \mid P(n)\} > n$
 b. $\llbracket \text{fewer than} \rrbracket = \lambda n \lambda P_{\langle d,t \rangle} . \max\{n \mid P(n)\} < n$

For Class B modifiers, it is probably already obvious that I will need to provide them with denotations that are distinct from those proposed by Nouwen, because my de-Fregean semantics of unmodified numerals is identical to Nouwen’s semantics for modified numerals with Class B maximizing modifiers like *at most*. This is arguably a good result, since one of the challenges for Nouwen’s analysis derived from the fact that *at most three* ended up producing meanings that were identical to the meanings of sentences containing bare *three*. As we saw, Nouwen used this result to invoke blocking principles and last-resort modalized interpretations to explain the properties of sentences containing *at most three*, but this approach led to the problems with modals that we saw in Section 2.2. Instead, I will propose that Class B modifiers are just like Class A modifiers except that they introduce partial orderings rather than total orderings:

- (42) a. $\llbracket \text{at least} \rrbracket = \lambda m \lambda P_{\langle d,t \rangle} . \max\{n \mid P(n)\} \geq m$
 b. $\llbracket \text{at most} \rrbracket = \lambda m \lambda P_{\langle d,t \rangle} . \max\{n \mid P(n)\} \leq m$

In fact, exactly this analysis of Class B modifiers is considered by Nouwen himself, but rejected as inadequate. (It is also used by [Büring 2008](#), though only as a placeholder for the alternatives-based account that he eventually settles on.) I will explain and respond to Nouwen’s criticisms presently; first, I will show how this semantic analysis of modified numerals, together with the semantics for bare numerals presented in the previous section, supports a pragmatic account of the facts we considered in Section 2.

4 A neo-Gricean pragmatics for modified numerals

4.1 Ignorance implicatures

My account of the ignorance implications associated with Class B modifiers is essentially the same as the pragmatic accounts proposed by [Büring 2008](#) and [Cummins & Katsos 2010](#): they arise as conversational implicatures from the utterance of a sentence whose semantic content is less informative than potential alternatives. In particular, they arise from reasoning involving the Maxim of Quantity, whereby the utterance of a weaker alternative indicates that the speaker does not know whether the stronger alternatives hold. (See also [Rett 2014](#), who develops a pragmatic analysis that combines Quantity and Manner reasoning.) Where my analysis moves beyond those of Büring and Cummins & Katsos is in providing a concrete account, rooted in a general semantics for number words, of what the alternatives to Class B modified numerals are, and why.

Let me first lay out my general assumptions about how ignorance implicatures are generated. I will adopt the model of quantity implicature calculation presented in [Sauerland 2004](#) (cf. [Horn 1972](#), [Gazdar 1977](#)), though other options would work just as well, including the “grammatical” analyses in [Chierchia 2004, 2006](#), [Fox 2007](#) and [Chierchia, Fox & Spector 2012](#) among others, or the game-theoretic analysis developed in [Franke 2011](#). In Sauerland’s system, the set of “primary” implicatures associated with an utterance of a particular sentence ϕ is the set defined in (43), where K is an (impersonal) epistemic certainty predicate, and $\text{ALT}(\phi)$ are the alternatives of ϕ , about which I will say more below.

- (43) $\{\neg K(\psi) \mid \psi \in \text{ALT}(\phi) \wedge \psi \Rightarrow \phi \wedge \phi \not\Rightarrow \psi\}$

Primary implicatures are propositions of the form ‘It is not known that ψ ’ where ϕ is an alternative of ψ that asymmetrically entails it; that is, they are ignorance implications. In some cases, primary implicatures can be strengthened to “secondary” implicatures as described in (44).

- (44) If $\neg K(\psi)$ is a primary implicature of ϕ and $K(\neg\psi)$ is consistent with the conjunction of ϕ and all primary implicatures of ϕ , then $K(\neg\psi)$ is a secondary implicature of ϕ .

Secondary implicatures do not involve ignorance, and instead express certainty that stronger alternatives do not hold: they are upper-bounding (scalar) implicatures. In order to show that the ignorance implications of Class B numerals arise pragmatically, then, I need to show that Class B modified numerals give rise to (appropriate) primary implicatures but not to secondary implicatures, and that Class A modified numerals do not give rise to primary or secondary implicatures. And the way I do that is in terms of the theory of alternatives for utterances involving numerals.

Along with Sauerland and others in the neo-Gricean tradition, I adopt the view that the alternatives relevant for implicature calculation are derived by substitution of expressions that form a quantitative or “Horn” scale (see e.g. Horn 1972, Gazdar 1977, Hirschberg 1985, Matsumoto 1995, Sauerland 2004). In virtually all work on the semantics and pragmatics of numerals since Horn 1972, the set of basic number words is assumed to form a Horn scale, an assumption that is indeed necessary to derive two-sided utterance content if sentences containing numerals “assert lower-boundedness”, as Horn 1972 claims in the quotation at the beginning of the previous section.

However, as several authors have pointed out, the very same assumptions that provide the desired result for sentences with unmodified numerals lead to problems with sentences containing modified numerals (see e.g. Krifka 1998, Fox & Hackl 2007, Meyr 2013). If numerals form Horn scales, then each of the following sentences should have alternatives in which *six* is replaced by other numerals:

- (45) a. This airplane has six emergency exits.
b. This airplane has at least six emergency exits.
c. This airplane has more than six emergency exits.

But if this is correct, we now need to explain not only why (45b) but not (45c) generates uncertainty implicatures, but also why neither (45b) nor

(45c) have upper-bounding implicatures. The literature contains different proposals for how to deal with this problem, which range from stipulating differences between modifiers in the ways that they interact with alternatives (Krifka 1998, Geurts & Nouwen 2007) to exploiting the underlying structure of quantity scales (Fox & Hackl 2007), but each account has its own empirical and theoretical shortcomings (see Meyr 2013 for discussion).

Here I want to propose a different kind of explanation for the absence of upper-bounding implicatures in (45b-c), which will also turn out to provide the basis for an explanation of why only Class B modifiers give rise to ignorance implications: *numerals do not form Horn scales*. Clearly, if the de-Fregean semantics of bare numerals presented in the previous section is correct, then is it not necessary to assume that numerals form Horn scales to derive an upper-bounding implication in examples like (46a), because this is already part of the semantic content of the sentence. But more than that, it arguably *follows* from the de-Fregean semantic analysis that numerals do not form Horn scales, since numerals *qua* de-Fregean generalized quantifiers are non-monotonic relative to other numerals, and Horn scales are generally taken to consist of items that support a quantitative ordering (Horn 1972, Gazdar 1977, Hirschberg 1985, Matsumoto 1995).

Instead, I propose that the Horn scales that are relevant for the calculation of implicatures of sentences containing numerals consist of all the modified and unmodified variants of those particular numerals.⁴ On this view, the alternatives that feed into the calculation of the implicatures associated with an utterance of (46), where “___” is filled by some form of the numeral *six*, are the sentences in (46a-c) (or the propositions they express) and other Class A/B variants.

- (46) This airplane has ___ emergency exits.
- a. This airplane has six emergency exits.
 - b. This airplane has more/fewer than six emergency exits.
 - c. This airplane as at least/most six emergency exits.

⁴ Schwarz 2013 resists this move, pointing out that *at least/most* have different distributions from *more/fewer than*. This is correct, and raises questions about the generality of the analysis proposed here, which I will address in the conclusion. But difference of distribution is not an impediment to forming a Horn scale. Adnominal *all* and *some* have different distributions (e.g., the former can float; the latter cannot), and yet it is clear that they can be members of the same Horn scales.

In the case of an utterance of a sentence with an unmodified numeral, such as (46a), no primary implicatures are generated because none of the alternatives in (46b–c) entail the utterance. The same holds for an utterance of a sentence with a Class A modified numeral such as *more/fewer than six* in (46b), and so we explain why Class A modifiers do not generate uncertainty implications: they do not have (relevant) alternatives, and so they do not generate primary/epistemic implicatures. They do not give rise to secondary/upper-bounding implicatures for exactly the same reason.

In the case of an utterance of a sentence with a Class B modified numeral, however, such as *at least six* or *at most six* in (46c), the situation is different: the *at least* sentence is asymmetrically entailed by the bare numeral alternative and the *more than* alternative, and the *at most* sentence is asymmetrically entailed by the bare numeral alternative and the *fewer than* alternative. The primary implicatures of the two variants of (46c) are therefore as follows:

- (47) a. $\{\neg K(= 6), \neg K(> 6)\}$ *at least six*
 b. $\{\neg K(= 6), \neg K(< 6)\}$ *at most six*

An utterance of *This airplane has at least (most) six emergency exits* thus implicates that it is not known whether the airplane has (exactly) six emergency exits and it is not known whether the airplane has more (fewer) than six emergency exits. Moreover, strengthening either primary implicature to its $K(\neg\phi)$ variant would contradict the conjunction of the assertion plus the other primary implicature, so we derive no secondary implicatures. Instead, we derive all and only the correct ignorance inferences of Class B modified numerals as quantity implicatures.⁵

⁵ Schwarz & Shimoyama (2011) present a pragmatic analysis of uncertainty implications of *wa*-marked measure phrases in Japanese sentences such as (i) that has essentially the same structure as the analysis of Class B modifiers that I am proposing here, but differs in a key semantic respect that makes it problematic for the analysis of modified numerals.

- (i) Taro-wa doitu-ni too-ka(-kan)-wa taiziasimasi-ta.
 Taro-TOP Germany-DAT ten-day(-for)-WA stay-PAST
 ‘Taro stayed in Germany for at least ten days.’

The crucial semantic difference between Schwarz & Shimoyama’s analysis and mine is that the two-sided alternative to (i) is not based on the semantics of the measure phrase, but rather on the semantics of the implicit degree predicate that it saturates, which relates (maximal) events to their durations. Schwarz & Shimoyama take this predicate to denote an equality relation (‘The duration of $e = d$ ’), and propose that the function of *-wa* is to turn it into a partial ordering (‘The duration of $e \geq d$ ’). (So semantically, *-wa* composes not with the measure phrase, but with this implicit predicate, contrary to appearance.)

4.2 Interactions with modals

Let us now turn to the interactions of Class B modifiers with modals. First, it should be clear that we have no problem with epistemic modals: both bare and modified numerals are degree quantifiers, and (for whatever reason) cannot take scope over epistemic modals. This was a bit of a puzzle for Nouwen because his account of uncertainty inferences crucially relied on inserting an epistemic modal inside the scope of a Class B modifier. Since the current account does not rely on the presence of an epistemic modal to generate uncertainty inferences, this problem disappears. The real test, then, involves the pattern of interactions with root modals. In the next two sections, I look at the interactions with universal and existential root modals, respectively, showing how the analysis can derive the range of attested interpretations. The interactions with root modals are accounted for straightforwardly, but as we will see, the interactions with existential modals are somewhat more complicated, in particular the interaction of existential modals and *at most*.

4.2.1 Universal modals

Let us begin with universal modals and minimizing Class B modifiers. Consider the example in (48), which is predicted to have the two interpretations in (48a–b), depending on whether the numeral scopes above or below the modal.

(48) You are required to register for at least three classes.

- a. $\max\{n \mid \Box[\exists x[\text{reg}(x)(\text{you}) \wedge \text{classes}(x) \wedge \#(x) = n]]\} \geq 3$
- b. $\Box[\max\{n \mid \exists x[\text{reg}(x)(\text{you}) \wedge \text{classes}(x) \wedge \#(x) = n]\} \geq 3]$

Looking first at truth-conditional content, (48a) is true just in case the maximum n such that in every deontically accessible world there is registration

It should be clear that this analysis will not work for numerals, at least not given the assumption that the individual argument of the nominal that the numeral composes with is existentially bound. As we saw in Section 3.1, even if we assume that the degree predicate involved in counting expresses an equality relation that relates a (plural) individual to its count ('The number of $x = d$ '), existential quantification over the individual argument delivers lower-bounded truth conditions for the sentence as a whole. We could introduce two-sided truth conditions by maximizing over the individual argument, but this move has non-trivial semantic consequences, as discussed in great detail in Brasoveanu 2013. Alternatively, we could assume that it is the numeral/measure phrase itself that introduces two-sided content, as I have argued in this paper, and treat *-wa* on a par with other Class B numeral modifiers.

in at least n classes is at least three, and (48b) is true just in case every deontically accessible world is one that involves registration in at least three classes. These are equivalent truth conditions — and are indeed the truth conditions of this sentence — but the two logical forms give rise to distinct implicatures, as pointed out by [Büring 2008](#).

The crucial alternatives for (48a) are sentences in which bare *three* or comparative *more than three* scope over the modal, the meanings of which I abbreviate as $\max(\Box) = 3$ and $\max(\Box) > 3$, respectively. Recall that when a numeral outscopes a necessity modal, *max* returns the “deontic lower bound”: the maximum value that all deontically accessible worlds agree on. The alternatives to (48a) asymmetrically entail it, so we get the primary implicatures in (49), neither of which can be strengthened without contradicting the combination of the other plus the assertion.

(49) $\{\neg K(\max(\Box) = 3), \neg K(\max(\Box) > 3)\}$

More generally, whenever a Class B modified numeral outscopes a necessity modal or a possibility modal (as we will shortly see), we derive uncertainty implicatures. This is what [Büring 2008](#) calls the “speaker insecurity” reading of a sentence like (48).

The sentence in (48) also has what Büring calls an “authoritative” reading, which is heard clearly in an example like (50).

(50) Your password must contain at least three numeric characters.

As Büring points out, this reading is derived when the numeral takes scope below the modal, as in (48b). The relevant alternatives in this case are $\Box(\max = 3)$ and $\Box(\max > 3)$, both of which asymmetrically entail the corresponding narrow scope logical form for the Class B modifier, so we derive the primary implicatures in (51).

(51) $\{\neg K(\Box(\max = 3)), \neg K(\Box(\max > 3))\}$

Unlike what we saw above, the primary implicatures can be strengthened to the corresponding secondary implicatures:

(52) $\{K(\neg\Box(\max = 3)), K(\neg\Box(\max > 3))\}$

Given that the assertion commits the speaker to certainty that the requirements stipulate enrollment in three or more classes, the implicatures in (52) entail that registration in more than three classes is allowed and registration in exactly three classes is allowed. This is the authoritative reading.

Now consider maximizing Class B modifiers with a universal deontic modal:

- (53) You are required to register for at most three classes.
- a. $\max\{n \mid \Box[\exists x[\text{reg}(x)(\text{you}) \wedge \text{classes}(x) \wedge \#(x) = n]]\} \leq 3$
 - b. $\Box[\max\{n \mid \exists x[\text{reg}(x)(\text{you}) \wedge \text{classes}(x) \wedge \#(x) = n]\} \leq 3]$

The proposition in (53a) says that the maximal n such that in every accessible world there is registration in at least n classes is less than or equal to three. This reading allows for enrollment in more than three classes, and as we saw above, it should give rise to an uncertainty implicature. The sentence in (54), adapted from a sentence that I discovered on a course syllabus, clearly has this kind of reading.

- (54) You will be required to write at most four short critical review papers on articles assigned in class (though you may write more if you wish), with the exact number to be determined by the end of the fifth week.

The proposition in (53b) says that every deontically accessible world is such that there is registration in three or fewer classes. This is the most salient interpretation of (53), and is understood authoritatively for the same reasons that we saw above. The proposition in (53b) gives rise to the primary implicatures in (55a), which can be strengthened to the secondary implicatures in (55b); together with the assertion, these implicatures entail that registration in exactly three classes is allowed and registration in fewer than three classes is allowed.

- (55) a. $\{\neg K(\Box(\max = 3)), \neg K(\Box(\max < 3))\}$
 b. $\{K(\neg\Box(\max = 3)), K(\neg\Box(\max < 3))\}$

The following naturally occurring example quite clearly illustrates the authoritative reading of *at most n* under a universal deontic modal:

- (56) To comply with the US's bigamy laws, foreign visitors with multiple wives are required to bring at most one wife into the US.

4.2.2 Existential modals

Let us now turn to minimizing Class B modifiers and existential modals:

(57) You are allowed to register for at least three classes.

a. $\max\{n \mid \Diamond[\exists x[\text{reg}(x)(\text{you}) \wedge \text{classes}(x) \wedge \#(x) = n]]\} \geq 3$

b. $\Diamond[\max\{n \mid \exists x[\text{reg}(x)(\text{you}) \wedge \text{classes}(x) \wedge \#(x) = n]\} \geq 3]$

The proposition in (57a) says that the maximum number of allowable, registered-for classes is three or more, which does not entail that it is forbidden to register in one or two classes. (Recall that this was a problem for Nouwen’s analysis of this example.) And since the numeral outscopes the modal, it should give rise to uncertainty implicatures. (Examples with existential modals also introduce secondary/upper-bounding implicature based on alternatives with universal modals, which I will ignore in what follows.) The example in (58), which is taken from an anti-government website on which the site’s author documents his beliefs about U.S. government cooperation with invading grey aliens, provides a nice illustration of this reading.

(58) The greys were also allowed to build at least 3 underground bases in which to operate out of.

Given the rest of the context surrounding this example, it is clear that the author is certain only that there are at least three alien bases, and is uncertain whether the aliens might have been allowed to build more.

When the numeral takes scope below the modal, as in (57b), the resulting truth conditions are weak but not problematic: there’s a possible world in which at least three classes are taken. The predicted epistemic implicatures, however, appear to be incorrect. As shown in (59), given what I have said so far, the narrow-scope numeral interpretation should implicate that it is not known whether enrollment in three classes is allowed and it is not known whether enrollment in more than three classes is allowed, but both of these implicatures contradict the assertion.

(59) $\{\neg K(\Diamond(\max = 3)), \neg K(\Diamond(\max > 3))\}$

We might assume that the conflict between implicatures and assertion here effectively blocks a narrow scope parse, leaving only the speaker uncertainty meaning associated with (57a) as a viable reading for sentences involving possibility modals (cf. Büring 2008). I would like to argue, however, that the narrow-scope parse is possible (and is attested), but that it gives rise to a different set of implicatures, which in fact constitute an authoritative reading. To see why, however, let us turn to the case of maximizing Class B

modifiers and existential modals, where the relevant meaning emerges even more clearly.

According to the analysis I have presented here, (6o) should allow two parses, corresponding to the readings in (60a-b).

(6o) You are allowed to register for at most three classes.

- a. $\max\{n \mid \diamond[\exists x[\text{reg}(x)(\text{you}) \wedge \text{classes}(x) \wedge \#(x) = n]]\} \leq 3$
- b. $\diamond[\max\{n \mid \exists x[\text{reg}(x)(\text{you}) \wedge \text{classes}(x) \wedge \#(x) = n]\} \leq 3]$

When the numeral takes wide scope over the modal, as in (60a), we predict truth conditions that forbid enrollment in more than three classes, and uncertainty about the actual cap on the number of courses. This is not the most salient reading of (6o), but it is certainly a possible reading of this type of sentence, as shown by (61).

(61) Students are allowed to drop at most three of their classes, but I don't know the exact number. Maybe they can only drop two.

The more salient reading of (6o) however is one that also forbids enrollment in more than three classes, but lacks uncertainty: it is an authoritative reading. The narrow-scope interpretation of the modified numeral in (60b) does not look like a very good candidate to deliver this meaning, however, for two reasons.

First, the truth conditions appear to be too weak, as pointed out already by Geurts & Nouwen 2007 and Nouwen 2010. The proposition in (60b) says merely that there is a deontically accessible world in which there's registration in three or fewer classes, which does not rule out registration in more classes. Yet (62a) sounds like a contradiction, unlike its variants with the Class A modifier *fewer than* or with an unmodified numeral:

- (62) a. Third-year students are allowed to register for at most three classes, #but they may register for more if they want to.
- b. Third-year students are allowed to register for (fewer than) three classes, but they may register for more if they want to.

It is precisely because of the contrast between (62a) and (62b) that Nouwen 2010 rejects the semantic analysis of Class B numerals that I have proposed here.

The second apparent problem for the putative narrow scope numeral interpretation of (6o) in (60b) is the same one that we ran into with the

corresponding narrow-scope reading of *at least* in (57b): we derive the wrong implicatures. The predicted primary implicatures of (60b) ought to be the ones in (63): it is not known whether registration in exactly three classes is allowed, and it is not known whether registration in fewer than three classes is allowed.

$$(63) \quad \{\neg K(\diamond(max = 3)), \neg K(\diamond(max < 3))\}$$

But these are obviously incompatible with an authoritative understanding of (60), and so, together with Nouwen’s observation about the weakness of the narrow scope *at most* truth conditions, appear to present a significant problem for my “naïve” extension of the de-Fregean semantics to Class B modifiers.

In fact, the semantic analysis I have proposed is able to derive precisely the correct implicatures, provided we adopt a proposal for the computation of alternatives that has been introduced independently in the literature to account for the so-called “free choice” interpretation of disjunction under modals, which shows a similar pattern of behavior to what we have seen here. (I am grateful to Michael Franke for suggesting this line of explanation to me.) The free choice interpretation of disjunction is illustrated in (64).

$$(64) \quad \text{Kim is allowed to stay home on Mondays or Fridays.}$$

Among the implications associated with an utterance (64) are the implications that Kim is allowed to stay home on Mondays and that Kim is allowed to stay home on Fridays; these, together with the assertion, constitute the free-choice interpretation. (There is also an exclusivity implication — Kim cannot stay home on both Mondays and Fridays — that I will ignore in what follows.) However, if the alternatives of (64) include propositions based on the individual disjuncts, as it is necessary to assume in order to derive uncertainty implicatures of simple disjunctions (see e.g. Sauerland 2004, Fox 2007, Franke 2011), then we derive the primary implicatures in (65), which are incompatible with the free-choice interpretation.

$$(65) \quad \{\neg K(\diamond(m)), \neg K(\diamond(f))\}$$

There is a family of solutions to this problem in the literature which differ in implementation but agree in the general idea that, in addition to the “simple” alternatives that give rise to the incorrect implicatures in (65), it is possible to generate a set of “exhaustified” alternatives, which deliver the primary implicatures in (66a) (Kratzer & Shimoyama 2002, Fox 2007, Franke

2011): it is not known whether Kim is only allowed to stay home on Mondays and it is not known whether Kim is only allowed to stay home on Fridays.

- (66) a. $\{\neg K(\diamond(m) \wedge \neg \diamond(f)), \neg K(\diamond(f) \wedge \neg \diamond(m))\}$
 b. $\{K\neg(\diamond(m) \wedge \neg \diamond(f)), K\neg(\diamond(f) \wedge \neg \diamond(m))\}$

The primary implicatures in (66a) can be further strengthened to the secondary implicatures in (66b): it is known that it is not the case that Kim can only stay home on Monday, and it is known that it is not the case that Kim can only stay home on Friday. Together with the assertion, these implicatures entail that Kim is allowed to stay home on Monday and that Kim is allowed to stay home on Friday, which are precisely the free choice inferences that we wanted to derive.

Returning to the case of modified numerals, once we introduce exhausted alternatives, the primary implicatures based on the narrow-scope numeral interpretation in (60b) are the ones shown in (67a), which can be strengthened to the secondary implicatures in (67b), just as we saw with the case of disjunction.

- (67) a. $\{\neg K(\diamond(max = 3) \wedge \neg \diamond(max < 3)),$
 $\neg K(\diamond(max < 3) \wedge \neg \diamond(max = 3))\}$
 b. $\{K\neg(\diamond(max = 3) \wedge \neg \diamond(max < 3)),$
 $K\neg(\diamond(max < 3) \wedge \neg \diamond(max = 3))\}$

The implicatures in (67b), together with the semantic content of (60b), entail that enrollment in three classes is allowed and that enrollment in fewer than three classes is allowed; that is, that enrollment in zero, one, two or three classes are all consistent with the rules. Allowing for exhaustification of alternatives thus derives a core part of what the authoritative reading of *at most* in (60) conveys. However, as we saw above, the authoritative reading also conveys that enrollment in more than three classes is not allowed, and since the input to the pragmatic analysis described above is a logical form in which the modified numeral takes narrow scope relative to the possibility modal, the problem of too-weak truth conditions remains. The combination of narrow scope semantic content plus quantity implicatures gives authoritative permission for enrollment in zero to three classes, but does not rule out enrollment in more than three classes, so it appears that we still lack an explanation for the fact that (62a) (but not (62b)) sounds like a contradiction.

I would like to suggest that the upper bound that we hear in examples like (62a) does not come from the semantics, but instead arises from additional

pragmatic reasoning, in this case about what is permitted, given what is conveyed by the semantic content plus the quantity implicatures. As noted by Sauerland 2004, in order to license computation of secondary implicatures in the first place, we must assume that the speaker is knowledgeable about the question under discussion. If the question under discussion is what the rules say about the number of classes that can be taken, if the knowledgeable speaker tells us that the rules permit registration in zero to three (the conjunction of the truth conditions plus the quantity implicatures), then we should assume that this is all that the rules allow. Considering higher values would imply that the speaker either didn’t tell us all that she has authoritative knowledge about, and so is not communicating in accord with pragmatic norms about the issuing of permission, or that she wasn’t knowledgeable in the first place. Both cases are incompatible with the quantity reasoning necessary to derive the authoritative meaning, so the conflict between the first and second sentences in (62a), on this view, is not at the level of truth conditions, but at the level of assumptions about the speaker and what her role in the communicative exchange must be like in order to support the pragmatic reasoning that leads to the authoritative meaning. The reason why there is no conflict in the case of a narrow-scope Class A modified numeral or unmodified numeral is that the utterance of such sentences do not give rise to quantity reasoning in the first place, since on the theory I have proposed here they do not have stronger alternatives. Paradoxically, then, it is by using the weakest semantic configuration — *at most* (rather than *fewer than* or the bare numeral) plus narrow scope — that the speaker is able to make the strongest utterance, since only this configuration licenses the quantity calculation that leads to the authoritative reading.⁶

There are a couple of pieces of evidence in favor of this explanation, and against an account that attempts to “hard-wire” the special behavior of *at most* in some way or other. First, if the reasoning is correct, then we should see the same effect when we make the quantity implicatures explicit in the semantic content. (68), which sounds more like (62a) than the variant of (62b) with the bare numeral, indicates that this is the case.

⁶ The reasoning outlined here also helps us understand why we don’t get an implicature based on the modal ($K \neg \Box(\text{max} \leq 3)$), which we might otherwise expect given the Sauerland-style approach to implicature calculation that I have adopted (see in particular Sauerland 2004, p. 378). Such an implicature would eliminate the upper bound and so would be incompatible with the assumption that what the knowledgeable authority does not allow is not allowed.

- (68) #Third-year students are allowed to register for zero to three classes, but they may register for more if they want to.

Second, it is possible to find examples in which *at most* clearly has scope below an existential modal, and the meaning derived above (truth conditions plus authoritative implicatures) is exactly what we want. The last sentence in (69), taken from a job advertisement on the internet, is one such example:

- (69) The physical demands described here are representative of those that must be met by an employee to successfully perform the essential functions of this job. Reasonable accommodations may be made to enable individuals with disabilities to perform the essential functions. Must be able to lift at most 70lbs.

There are in principle three different ways that this sentence could be interpreted, depending on whether the modified measure phrase (which I treat analogously to a numeral) has widest, intermediate, or narrowest scope:

- (70) a. $\max\{n \mid \Box[\Diamond[\text{lift}(n)]]\} \leq 70\text{lbs}$
 b. $\Box[\max\{n \mid \Diamond[\text{lift}(n)]\} \leq 70\text{lbs}]$
 c. $\Box[\Diamond[\max\{n \mid \text{lift}(n)\} \leq 70\text{lbs}]]$

The reading in (70b) is clearly wrong, because it rules out potential employees who have an ability to lift more than seventy pounds, which is not how the advertisement is understood: nobody will be turned away (only) because they are too strong. The reading in (70a) does not rule out people who can lift more than seventy pounds (for the reasons we have seen), but it should generate an uncertainty implicature about the actual weight-lifting abilities that are required, which is also not how the advertisement is understood. Instead, the advertisement is understood to convey the information in (70c) — that it is required that it be within an employee’s ability to lift a maximal weight that falls in the open interval bounded below by zero and closed on the top by seventy — *plus* the authoritative implicatures that are derived according to the algorithm described above: it is not only required that it be within an employee’s ability to lift less than seventy pounds, and it is not only required that it be within an employee’s ability to lift exactly seventy pounds. The difference between this case and one in which the existential modal is not embedded (e.g., *This employee can lift at most 70lbs*, in which the upper bound inference returns) is that the question under discussion is not the employee’s abilities, but rather the job requirements, and it is possible to

convey maximal information about the job requirements without introducing an upper bound on abilities.

Returning to the case of *at least*, the same reasoning that derives authoritative implicatures for *at most* under an existential modal will derive corresponding authoritative implicatures for *at least* under an existential modal. The secondary implicatures associated with the narrow scope interpretation of *at least three* in (71a), for example, are shown in (71b).

(71) You are allowed to register for at least three classes.

- a. $\Diamond[\max\{n \mid \exists x[\text{reg}(x)(\text{you}) \wedge \text{classes}(x) \wedge \#(x) = n]\} \geq 3]$
- b. $\{K\neg(\Diamond(\max = 3) \wedge \neg\Diamond(\max > 3)),$
 $K\neg(\Diamond(\max > 3) \wedge \neg\Diamond(\max = 3))\}$

The combination of the semantic and pragmatic content here says that registration in three classes is acceptable and registration in more than three classes is acceptable, and moreover doesn’t rule out registration in fewer classes. This kind of reading does not provide particularly useful information about what’s allowed, and so is likely to be highly dispreferred (cf. Büring 2008). But the following naturally occurring example suggests that it is not impossible:

(72) Previously in Germany, students were allowed to take at least five years to complete the Magister’s diploma, the basic university degree. But now, Germany has adopted the Anglo-Saxon style of bachelor’s and master’s degrees. The bachelor’s degree is designed to take three years to complete; the master’s, a further two years.

Here there is no uncertainty about the number of years German students were allowed to take to finish their diplomas: it was five *or more*, and we know that more than five was an option because the former system is being contrasted to the new one, in which students must finish within five years.⁷

⁷ Doris Penka (p.c.) suggests that the apparent authoritative reading here — or rather, the apparent absence of an ignorance implicature — could be explained instead in terms of the interaction between the superlative modifier and the bare plural subject, on analogy to the interaction between superlatives and universally quantified subjects (see e.g. Schwarz & Shimoyama 2011) in examples like (ia).

- (i) a. Every student registered for at least three classes.
- b. $\forall x[\text{student}(x) \rightarrow \max\{n \mid \exists y[\text{reg}(y)(x) \wedge \text{class}(y) \wedge \#(y) = n]\} \geq 3]$

5 Concluding remarks

In this paper, I have presented a “de-Fregean” semantics for modified and unmodified numerals as generalized quantifiers over degrees, in which bare numerals, Class A modified numerals and Class B modified numerals all have essentially the same semantic analysis, differing only in the kind of ordering relation they introduce over the unique (maximal) degree that satisfies their scope. The denotations in (73) illustrate the basic pattern for unmodified *six*, Class A modified *more than six*, and Class B modified *at least six*.

- (73) a. $\llbracket \text{six} \rrbracket = \lambda P_{\langle d,t \rangle}. \max\{n \mid P(n)\} = 6$
 b. $\llbracket \text{more than six} \rrbracket = \lambda P_{\langle d,t \rangle}. \max\{n \mid P(n)\} > 6$
 c. $\llbracket \text{at least six} \rrbracket = \lambda P_{\langle d,t \rangle}. \max\{n \mid P(n)\} \geq 6$

I have shown that this account can derive the uncertainty implications associated with Class B modifiers as Quantity implicatures that arise from the fact that a sentence with a Class B modified numeral n is weaker than the relevant alternatives, which I argued are the corresponding sentences with unmodified n or Class A modified n . I implemented my analysis in terms of a fully neo-Gricean theory of implicature calculation, though the core proposals are compatible with other approaches. But, crucially, the constraint that numerals do not form Horn scales — or, more generally, do not constitute alternatives for other numerals — must be retained. I suggested that this constraint is related to the fact that, given a de-Fregean semantics, bare numerals are non-monotonic, which in turn points to a semantic basis for the determination of the alternatives relevant to implicature calculation, rather than a structural basis, as argued in recent work by [Katzir 2007](#).

There are a number of respects in which the proposals I have made here are incomplete, which should be pointed out. First, there are individual differences between the modifiers in the Class A and Class B groups having

Assuming that the degree quantifier takes scope below the universal quantifier, as shown in (ib) ([Kennedy 1999](#), [Heim 2000](#)), the primary implicatures should be that it is not known that every student is such that she registered for exactly three classes and that it is not known that every student is such that she registered for more than three classes. But these implicatures are consistent with knowing for every student exactly how many classes she took, as long as the resulting distribution gives back a range of enrollment numbers whose lowest value is three. Whether this treatment can be extended to (72) depends on whether it is plausible to analyze the bare plural here as involving distributive quantification vs. as kind-denoting, and also on a careful consideration of the predicted meanings, in particular given the fact that *students* appears to have scope below the modal.

to do with, for example, whether they are compatible with a single degree satisfying their scope, or whether they require an interval (Nouwen 2008a, Schwarz, Buccola & Hamilton 2012, Rett 2014). A more fine-grained analysis of the lexical semantics of the various modifiers and the way their uses as numeral modifiers relate to their other uses as comparative or locative morphemes is necessary to provide a full account of these differences.

Second, I have said very little about how a degree semantics for modified and unmodified numerals interacts with quantification over the individual argument of the nominal, and in particular with issues of distributivity, cumulativity, etc. (see e.g. Krifka 1998, Brasoveanu 2013), other than to stipulate that the latter is bound by a default existential quantifier.

Finally, and most significantly, the proposals I have made here are surely insufficient as a full semantic account of the superlative modifiers *at least* and *at most* (and probably insufficient as an account of the adverbials *minimally* and *maximally* as well). The denotations that I gave for these modifiers in Section 3.2 treat them as expressions that map degrees (in particular, singular term numeral denotations) to generalized quantifiers over degrees, which leads to the prediction that they should combine only with degree-denoting expressions. But this prediction is wrong: it is a well-known feature of these expressions both that they can combine with a range of different categories, as shown in (74b–c). (I use *at most* for illustration, but similar examples can be constructed with *at least*.)

- (74) a. We should invite at most three linguists, not two.
b. We should invite at most Kim, Lee and Pat, not Mo as well.
c. We should invite at most some linguists, not some philosophers as well.

They can, moreover, be separated from the expression with which they are understood to associate:

- (75) a. We should at most invite three linguists.
b. We should at most invite Kim, Lee and Pat.
c. We should at most invite some linguists.
- (76) a. We should invite three linguists at most.
b. We should invite Kim, Lee and Pat at most.
c. We should invite some linguists at most.

This latter syntactic fact is reflected in the semantics by focus-sensitivity: in both split and non-split forms, focus determines what is understood to be the associate of the modifier.

- (77) a. We should at most invite THREE linguists,
not FOUR/??three PHILOSOPHERS.
b. We should at most invite three LINGUISTS,
not ??FOUR/three PHILOSOPHERS.
- (78) a. We should invite at most THREE linguists,
not FOUR/??three PHILOSOPHERS.
b. We should invite at most three LINGUISTS,
not ??FOUR/three PHILOSOPHERS.

These kinds of facts have led a number of authors to develop accounts of these modifiers within a focus/alternatives semantics (see e.g. [Krifka 1998](#), [Geurts & Nouwen 2007](#), [Cohen & Krifka 2010](#)), with recent work by [Coppock & Brochhagen 2013](#) geared specifically towards showing how this kind of approach, together with an inquisitive semantics, can capture the Class A/Class B distinction. Given that such an approach is likely to be independently necessary in order to account for focus-sensitivity, cross-categoriality, and split variants of modified numerals, it is worth asking whether this will ultimately make the degree-theoretic, de-Fregean analysis I have proposed here superfluous.

Although a full answer to this question goes beyond the scope of this paper, my guess is that both kinds of analysis will ultimately be necessary, and that a fully comprehensive treatment of modified and unmodified numerals will involve integrating the focus-sensitive and degree-theoretic approaches. The latter provides us with a compositional framework that reflects the growing evidence that modified and unmodified numerals are syntactic and semantic constituents that saturate a quantity position inside the nominal projection and can take scope independently of the nominal with which they compose on the surface (see e.g. [Hackl 2000](#), [Heim 2000](#), [Takahashi 2006](#), [Nouwen 2008b](#), [Kennedy & Stanley 2009](#), [Nouwen 2010](#), [Kennedy 2013](#)).

It also provides us with a simple account of minimal pairs like the following, where the (a) examples are strange because they (unlike the (b) examples) implicate uncertainty about how many sides a hexagon has:

- (79) a. #A hexagon has no more than ten sides.
b. A hexagon does not have more than ten sides.

- (80) a. #A hexagon has no fewer than four sides.
b. A hexagon does not have fewer than four sides.

In a fully compositional semantics of comparatives, *more/fewer than* are properly analyzed not as introducing the simple ordering relations $>$ and $<$, but as introducing a requirement for a positive difference in degree (see e.g. von Stechow 1984, Schwarzschild 2005, Alrenga & Kennedy 2014). The function of the negative differential expression *no* is to say that there is no positive difference (Alrenga & Kennedy 2014), so *no more than* is consistent with either $=$ or $<$, and *no fewer than* is consistent with either $=$ or $>$. The result is that *no more/fewer than* have the same ordering entailments as *at most/at least*, so it is no surprise, from a degree-theoretic perspective, that they show the Class B uncertainty pattern.⁸

Given these considerations, it seems that even an alternatives-based analysis of superlative modifiers needs to hypothesize such expressions to take on meanings that allow them to combine with numerals to derive generalized quantifiers over degrees. And indeed, this is the approach taken by Coppock & Brochhagen 2013, who show that the basic meanings they assume for *at most* and *at least* as propositional modifiers can be shifted into meanings that combine with a numeral, generating a generalized quantifier over degrees which introduces truth conditions that are essentially equivalent to what we get on the analysis of *at least/most n* that I have proposed here. What we also need to ask is whether it is also possible to go in the other direction: to move from a degree-based semantics and pragmatics for comparative and superlative modifiers — or from an even more basic semantics and pragmatics for comparative and superlative degree morphology — to an alternatives-based semantics and pragmatics that has the same breadth of coverage as Coppock & Brochhagen’s, and that derives the distributional differences between comparative and superlative modifiers.

⁸ What is a surprise is that unlike *at most/least* and other Class B modifiers, they can give rise to scalar bounding inferences, so that for instance *no more than ten* is often understood to convey *ten*. (This is not the reason for the unacceptability of (79a) and (80a), however: as we saw at the very beginning of the paper, it is perfectly acceptable to falsely assert that a hexagon has ten or four sides.) Nouwen 2008b concludes from this that *at most/least* and other Class B modifiers cannot mean the same thing as *no more/fewer than*, but the fact that *no more/fewer than* also introduce uncertainty inferences (a fact that Nouwen does not mention) shows that the situation is more complicated. I do not know what is going on here, though I suspect that the extra complexity of *no more/fewer than* — differential *no* is itself a degree quantifier (Alrenga & Kennedy 2014) — is significant; cf. Rett’s (2014) discussion of “measure phrase equatives” such as *as many as six*.

References

- Alrenga, Peter & Christopher Kennedy. 2014. *No more shall we part: Quantifiers in English comparatives*. *Natural Language Semantics* 22. 1–53. <http://dx.doi.org/10.1007/s11050-013-9099-4>.
- Brasoveanu, Adrian. 2013. Modified numerals as post-suppositions. *Journal of Semantics* 30(2). 155–209. <http://dx.doi.org/10.1093/jos/ffs003>.
- Breheny, Richard. 2008. A new look at the semantics and pragmatics of numerically quantified noun phrases. *Journal of Semantics* 25. 93–139. <http://dx.doi.org/10.1093/jos/ffm016>.
- Bultinck, Bert. 2005. *Numerous meanings: The meaning of English cardinals and the legacy of Paul Grice*. Amsterdam: Elsevier.
- Büring, Daniel. 2007. *More or less*. *Chicago Linguistics Society (CLS)* 43. 3–17.
- Büring, Daniel. 2008. The least *at least* can do. *West Coast Conference on Formal Linguistics (WCCFL)* 26. 114–120.
- Carston, Robyn. 1998. Informativeness, relevance and scalar implicature. In Robyn Carston & S. Uchida (eds.), *Relevance theory: Applications and implications*, 179–236. Amsterdam: John Benjamins.
- Chierchia, Gennaro. 2004. Scalar implicatures, polarity phenomena and the syntax/pragmatics interface. In Adriana Belletti (ed.), *Structures and beyond*. Oxford, UK: Oxford University Press.
- Chierchia, Gennaro. 2006. Broaden your views: Implicatures of domain widening and the “logicality” of language. *Linguistic Inquiry* 37(4). 535–590. <http://dx.doi.org/10.1162/ling.2006.37.4.535>.
- Chierchia, Gennaro, Danny Fox & Benjamin Spector. 2012. The grammatical view of scalar implicature and the relationship between semantics and pragmatics. In Claudia Maienborn, Klaus von Stechow & Paul Portner (eds.), *Semantics: An international handbook of natural language meaning*, vol. 3. Boston, MA: de Gruyter.
- Chung, Sandra & William A. Ladusaw. 2004. *Restriction and saturation*. Cambridge, MA: MIT Press.
- Cohen, Ariel & Manfred Krifka. 2010. Superlative quantifiers as modifiers of meta-speech acts. *The Baltic International Yearbook of Cognition, Logic and Communication* 6. 1–56. <http://dx.doi.org/10.4148/biyclc.v6io.1578>.
- Coppock, Elizabeth & Thomas Brochhagen. 2013. Raising and resolving issues with scalar modifiers. *Semantics and Pragmatics* 6(3). 1–57. <http://dx.doi.org/10.3765/sp.6.3>.

- Cresswell, M. J. 1976. The semantics of degree. In Barbara Partee (ed.), *Montague grammar*, 261–292. New York, NY: Academic Press.
- Cummins, Chris & Napoleon Katsos. 2010. Comparative and superlative quantifiers: Pragmatic effects of comparison type. *Journal of Semantics* 27. 271–305. <http://dx.doi.org/10.1093/jos/ffq006>.
- von Stechow, Kai & Sabine Iatridou. 2007. Anatomy of a modal construction. *Linguistic Inquiry* 38. 445–483. <http://dx.doi.org/10.1162/ling.2007.38.3.445>.
- Fox, Danny. 2007. Free choice disjunction and the theory of scalar implicatures. In Uli Sauerland & Penka Stateva (eds.), *Presupposition and implicature in compositional semantics*, 71–120. London: Palgrave MacMillan.
- Fox, Danny & Martin Hackl. 2007. The universal density of measurement. *Linguistics and Philosophy* 29. 157–186. <http://dx.doi.org/10.1007/s10988-006-9004-4>.
- Franke, Michael. 2011. Quantity implicatures, exhaustive interpretation, and rational conversation. *Semantics and Pragmatics* 4(1). 1–82. <http://dx.doi.org/10.3765/sp.4.1>.
- Frege, Gottlob. 1980 [1884]. *The foundations of arithmetic (Grundlagen der Arithmetik)*. Evanston, IL: Northwestern University Press.
- Gazdar, Gerald. 1977. *Implicature, presupposition and logical form*. Sussex, UK: University of Sussex PhD thesis.
- Geurts, Bart. 2006. Take *five*. In Svetlana Vogeteeler & Liliane Tasmowski (eds.), *Non-definiteness and plurality*, 311–329. Amsterdam: Benjamins. <http://dx.doi.org/10.1075/la.95.16geu>.
- Geurts, Bart & Rick Nouwen. 2007. *At least et al.*: The semantics of scalar modifiers. *Language* 83(3). 533–559. <http://dx.doi.org/10.1353/lan.2007.0115>.
- Hackl, Martin. 2000. *Comparative quantifiers*. Cambridge, MA: Massachusetts Institute of Technology PhD thesis.
- Heim, Irene. 2000. Degree operators and scope. *Semantics and Linguistic Theory (SALT)* 10. 40–64.
- Hirschberg, Julia. 1985. *A theory of scalar implicature*. Philadelphia, PA: University of Pennsylvania PhD thesis.
- Horn, Laurence. 1972. *On the semantic properties of logical operators in English*. Los Angeles, CA: University of California, Los Angeles PhD thesis.
- Horn, Laurence. 1992. The said and the unsaid. *Semantics And Linguistic Theory (SALT)* 2. 163–192.

- Huang, Yi Ting, Elizabeth Spelke & Jesse Snedeker. 2013. What exactly do numbers mean? *Language Learning and Development* 9. 105-129. <http://dx.doi.org/10.1080/15475441.2012.658731>.
- Hurewitz, Felicia, Anna Papafragou, Lila Gleitman & Rochel Gelman. 2007. Asymmetries in the acquisition of numbers and quantifiers. *Language Learning and Development* 2(2). 77-96. http://dx.doi.org/10.1207/s15473341llo202_1.
- Katzir, Roni. 2007. Structurally-defined alternatives. *Linguistics and Philosophy* 30. 669-690. <http://dx.doi.org/10.1007/s10988-008-9029-y>.
- Kennedy, Christopher. 1999. *Projecting the adjective: The syntax and semantics of gradability and comparison*. (1997 UCSC Ph.D Dissertation). New York, NY: Garland.
- Kennedy, Christopher. 2013. A scalar semantics for scalar readings of number words. In Ivano Caponigro & Carlo Cecchetto (eds.), *From grammar to meaning: The spontaneous logicality of language*, 172-200. Cambridge, UK: Cambridge University Press. <http://dx.doi.org/10.1017/cb09781139519328.010>.
- Kennedy, Christopher & Jason Stanley. 2009. On average. *Mind* 118(583-646). <http://dx.doi.org/10.1093/mind/fzp094>.
- Koenig, Jean-Pierre. 1991. Scalar predicates and negation: Punctual semantics and interval interpretations. *Chicago Linguistic Society (CLS)* 27. 140-155.
- Krasikova, Sveta. 2010. Comparing to what is possible and permitted. *Sinn and Bedeutung* 14. 256-272.
- Kratzer, Angelika & Junko Shimoyama. 2002. Indeterminate pronouns: The view from Japanese. 3.
- Krifka, Manfred. 1989. Nominal reference, temporal constitution and quantification in event semantics. In Renate Bartsch, Johann van Benthem & Peter van Emde Boas (eds.), *Semantics and contextual expression*, 75-115. Stanford, CA: CSLI Publications.
- Krifka, Manfred. 1998. At least some determiners aren't determiners. In Ken Turner (ed.), *The semantics/pragmatics interface from different points of view*, 257-291. Oxford, UK: Elsevier.
- Landman, Fred. 2003. Predicate argument mismatches and the adjectival theory of indefinites. In Martine Coene & Yves d'Hulst (eds.), *The syntax and semantics of noun phrases*, vol. 1 (From NP to DP), 211-238. Philadelphia, PA: John Benjamins. <http://dx.doi.org/10.1075/la.55.1olan>.
- Landman, Fred. 2004. *Indefinites and the type of sets*. Oxford, UK: Oxford University Press. <http://dx.doi.org/10.1002/9780470759318>.

- Marty, Paul, Emmanuel Chemla & Benjamin Spector. 2013. Interpreting numerals and scalar items under memory load. *Lingua* 133. 152–162. <http://dx.doi.org/10.1016/j.lingua.2013.03.006>.
- Matsumoto, Yo. 1995. The conversational condition on Horn scales. *Linguistics and Philosophy* 18. 21–60. <http://dx.doi.org/10.1007/bfo0984960>.
- Meyr, Clemens. 2013. Implicatures of modified numerals. In Ivano Caponigro & Carlo Cecchetto (eds.), *From grammar to meaning: The spontaneous logicity of language*, 139–171. Cambridge, UK: Cambridge University Press. <http://dx.doi.org/10.1017/cb09781139519328.009>.
- Musolino, Julien. 2004. The semantics and acquisition of number words: Integrating linguistic and developmental perspectives. *Cognition* 93(1). 1–41. <http://dx.doi.org/10.1016/j.cognition.2003.10.002>.
- Nouwen, Rick. 2008a. Directionality in numeral quantifiers: The case of *up to*. *Semantics and Linguistic Theory (SALT)* 18. 551–568.
- Nouwen, Rick. 2008b. Upper-bounded *no more*: The exhaustive interpretation of non-strict comparison. *Natural Language Semantics* 16. 271–295. <http://dx.doi.org/10.1007/s11050-008-9034-2>.
- Nouwen, Rick. 2010. Two kinds of modified numerals. *Semantics and Pragmatics* 3(3). 1–41. <http://dx.doi.org/10.3765/sp.3.3>.
- Noveck, Ira. 2001. When children are more logical than adults: Experimental investigations of scalar implicature. *Cognition* 78(2). 165–188. [http://dx.doi.org/10.1016/S0010-0277\(00\)00114-1](http://dx.doi.org/10.1016/S0010-0277(00)00114-1).
- Papafragou, Anna & Julien Musolino. 2003. Scalar implicatures: Experiments at the semantics-pragmatics interface. *Cognition* 86. 253–282. [http://dx.doi.org/10.1016/S0010-0277\(02\)00179-8](http://dx.doi.org/10.1016/S0010-0277(02)00179-8).
- Partee, Barbara. 1987. Noun phrase interpretation and type shifting principles. In Geroen Groenendijk, Dik de Jongh & Martin Stokhof (eds.), *Studies in discourse representation and the theory of generalized quantifiers*, 115–143. Dordrecht: Foris. <http://dx.doi.org/10.1002/9780470758335.ch15>.
- Rett, Jessica. 2014. Measure phrase equatives and modified numerals. *Journal of Semantics*. <http://dx.doi.org/10.1093/jos/ffu004>.
- Sadock, Jerrold. 1984. Whither radical pragmatics? In Deborah Schiffren (ed.), *Meaning, form and use in context: Linguistic applications*, 139–149. Washington, DC: Georgetown University Press.
- Salmon, Nathan. 1997. Wholes, parts, and numbers. *Noûs Supplement: Philosophical Perspectives*, 11, *Mind, Causation, and World* 31. 1–15. <http://dx.doi.org/10.1111/0029-4624.31.s11.1>.

- Sauerland, Uli. 2004. Scalar implicatures in complex sentences. *Linguistics and Philosophy* 27. 267-391. <http://dx.doi.org/10.1023/b:ling.0000023378.71748.db>.
- Scharten, Rosemarijn. 1997. *Exhaustive interpretation: A discourse-semantic account*. Nijmegen: Katholieke Universiteit Nijmegen PhD thesis.
- Schwarz, Bernhard. 2013. *At least* and quantity implicature: Choices and consequences. *Amsterdam Colloquium* 19. 187-194.
- Schwarz, Bernhard, Brian Buccola & Michael Hamilton. 2012. Two types of class B numeral modifiers: A reply to Nouwen 2010. *Semantics and Pragmatics* 5(1). 1-25. <http://dx.doi.org/10.3765/sp.5.1>.
- Schwarz, Bernhard & Junko Shimoyama. 2011. Negative islands and obviation by *wa* in Japanese degree questions. *Semantics and Linguistic Theory* 20. 702-719.
- Schwarzschild, Roger. 2005. Measure phrases as modifiers of adjectives. *Recherches Linguistiques de Vincennes* 35. 207-228. <http://dx.doi.org/10.4000/rlv.1401>.
- Schwarzschild, Roger & Karina Wilkinson. 2002. Quantifiers in comparatives: A semantics of degree based on intervals. *Natural Language Semantics* 10. 1-41. <http://dx.doi.org/10.1023/a:1015545424775>.
- von Stechow, Arnim. 1984. Comparing semantic theories of comparison. *Journal of Semantics* 3. 1-77. <http://dx.doi.org/10.1093/jos/3.1-2.1>.
- Takahashi, Shoichi. 2006. More than two quantifiers. *Natural Language Semantics* 14. 57-101. <http://dx.doi.org/10.1007/s11050-005-4534-9>.

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